Chapter 5 Linear Algebra

- Vectors in R^n : $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ or $X = (x_1, x_2, ..., x_n)$ (column or row form)
- dependence/independence: $V_1, ..., V_m \in R^n$
  - linearly dependent if there are $\alpha_1, ..., \alpha_m$ that are not all zero such that
  $$\alpha_1 V_1 + \alpha_2 V_2 + ... + \alpha_m V_m = 0$$
  - linearly independent if the only solution of above equation is $\alpha_1 = ... = \alpha_m = 0$
  - If $V_1, ..., V_m$ is linearly dependent, then at least one of them is a linear combination of the rest.
  - there are no more than n independent vectors in R^n
- Subspace of R^n
  - a subspace S is a subset that is closed under linear operations
    * $V, U \in S \implies aU + bV \in S$ for any constants a, b
    * let $a = b = 0 \implies 0 \in S$: any subspace must contain the origin
  - a subspace is spanned by a few vectors
  $$S = span\,\{V_1, ..., V_m\}$$
  - In R^2, only subspaces are lines passing the origin
  - In R^3, subspaces are lines and planes passing through the origin.
- Basis & coordinates
  - any subspace is generated by finite many vectors: $S = span\,\{V_1, ..., V_m\}$

    * if these generators $V_1, ..., V_m$ are linearly dependent, then one must be a linear combination of the rest.
    * Suppose V_m is a linear combination of the rest
    * V_m can be eliminated from the set of generators
    * i.e., $S = span\{V_1, ..., V_{m-1}\}$
    * So by reducing generators, we may end up a set of linearly independent generators
  - If $V_1, ..., V_m$ are linearly independent, we say $V = \{V_1, ..., V_m\}$ form a basis for S
  - a set $V = \{V_1, ..., V_n\}$ of n linearly independent in R^n is a basis of R^n
  - standard basis $E = \{e_1, e_2, ..., e_n\}$
  - For any vector X, there is a unique solution $\alpha_1, ..., \alpha_n$ for

$$X = \alpha_1 V_1 + \alpha_2 V_2 + ... + \alpha_n V_n$$

  - We call $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ the coordinate of X with respect to basis V

$$[X]_V = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

- Matrix and operations
  - $m \times n$ matrix

$$A = A_{nm} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

  - Column matrix form

$$A = (A_1 \ A_2 \ ... \ A_n), \quad A_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{pmatrix} \text{ is the } k\text{th column}$$

- matrix times a vector

$$AX = \begin{pmatrix} \sum_{k=1}^{n} a_{1k}x_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk}x_k \end{pmatrix}$$

- matrix A_{mn} times a matrix B_{np} is a matrix of dimension $m \times p$

- Inverse of matrix
  - a square matrix A_{nn} is invertible if there exists a C such that

$$AC = CA = I$$

  - C is called the inverse matrix of A, and denoted as $C = A^{-1}$
  - A is invertible iff columns of A are linearly independent
  - A is invertible iff the only solution for the following equation is $X = 0$:

$$AX = 0$$

  - A is invertible iff there is a unique solution for the following equation for any vector b :

$$AX = b$$

  - The solution is $X = A^{-1}b$
  - Inverse matrix algorithm by elementary row operations: $[A, I] \to [I, A^{-1}]$

- Changes of coordinates matrix
  - $V = \{V_1, ..., V_n\}$ is a basis
  - $[X]_V$ is the solution of

$$X = \alpha_1 V_1 + \alpha_2 V_2 + ... + \alpha_n V_n$$

  - In matrix notation

$$X = (V_1 \ V_2 \ ... \ V_n) \, [X]_V$$

- Let $V = [V_1 ... V_n]$ be the matrix whose columns are V_k. Then

$$X = V [X]_V$$
$$[X]_V = V^{-1} X$$

- Let $W = [W_1, ..., W_n]$ be another basis, then

$$[X]_W = W^{-1} X$$

- So we have the formula for changes of coordinate, the matrix for changes of coordinates:

$$[X]_W = W^{-1} V [X]_V$$

- Example 1: $V_1 = (1, 0, 0)\,,\ V_2 = (1, 1, 0)\,,\ V_3 = (1, 1, 1)\,.$

1. (a) show they form a basis

   (b) Find coordinate of vector $X = (0, 1, -1)$ with respect to the basis V

   (c) Show that $W_1 = (0, 1, 0)\,,\ W_2 = (0, 1, 1)\,,\ W_3 = (1, 0, 1)$ also form a basis

   (d) find *the* matrix representing changes of coordinate from $[X]_W$ to $[X]_V$ for all *any* X.

- Determinant $\det(A)$ of a square matrix
  - $\det(A)$ may be computed by row or column expansion, e.g., *ith* row :
  $$\det(A) = \sum_{k=1}^{n} (-1)^{1+k}\, a_{ik} \det(A_{ik})$$
  - A_{ik} is called a $(i,k)-$minor obtained from A by deleting row i and column k.
  - Similar column expansion
  - $\det(AB) = \det(A)\det(B)$
  - A is invertible iff $\det(A) \neq 0,\ \det(A^{-1}) = 1/\det A$
  - For triangle matrices A, $\det A =$ *product* of all diagonal entries
  - In particular, if $A = diag(\lambda_1, ..., \lambda_n)$, then $\det A = \lambda_1\lambda_2...\lambda_n$
  - elementary row operations for determinants

- Eigenvalues
  - λ is called an eigenvalue of A is

$$\det\left(A-\lambda I\right)=0$$

  - If λ is called an eigenvalue, then

$$AV=\lambda V$$

    has a non-trivial solution V.
  - We call V an eigenvector associated with λ.
  - Eigenvectors associated with different eigenvalues are always linearly independent.
  - diagonalization: if $n\times n$ matrix A has n distinct real eigenvalue, then it can be diagonalized
  - i.e., there is an invertible matrix T such that

$$T^{-1}AT=diag\left(\lambda_1,...,\lambda_n\right)$$

  - We call A is similar to $diag\left(\lambda_1,...,\lambda_n\right)$.
  - $T=\left(V_1,...V_m\right),\quad V_j$ is an eigenvector associated with λ_j

- Example 2 Find eigenvalues, eigenvectors, and then diagonalize:

$$A=\begin{pmatrix}1&2&-1\\0&3&-2\\0&2&-2\end{pmatrix}$$

- Complex eigenvalues
  - $\lambda_j = \alpha_j \pm i\beta_j,\ \ j = 1, 2, ..., m,$ are distinct complex eigenvalues of A
  - None of them is a conjugate of another. A has no other eigenvalue
  - complex eigenvector $V_j = V_j^{re} + iV_j^{im}$ associated with λ_j
  - $\left\{V_j^{re}, V_j^{im} : j = 1, 2, ..., m\right\}$ are linearly independent
  - Block matrix notation:

$$A = \begin{pmatrix} A_{11} & ... & A_{1q} \\ ... & \ddots & ... \\ A_{p1} & ... & A_{pq} \end{pmatrix}$$

  - Let $T = (V_1^{re}, V_1^{im}, V_2^{re}, V_2^{im}, ..., V_m^{re}, V_m^{im})$,then

$$T^{-1}AT = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & D_m \end{pmatrix}$$

  - where

$$D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

Canonical form

- Canonical form without repeated eigenvalues

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & D_1 & & \\ & & & & \ddots & \\ & & & & & D_l \end{pmatrix}$$

- Canonical form in general

$$T^{-1}AT = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_k \end{pmatrix} = B$$

– where B_j will be in one of the following two forms

$$(i)\quad \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}_{p\times p}, \quad (ii)\quad \begin{pmatrix} C_2 & I_2 & & \\ & C_2 & I_2 & \\ & & \ddots & I_2 \\ & & & C_2 \end{pmatrix}_{q\times q}$$

– where
$$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

– case (i) occurs when λ is a real eigenvalue with multiplicity p

– if $p = 1$, $B_j = (\lambda)$, a 1×1 matrix

– case (ii) occurs when $\lambda = \alpha + i\beta$ is a complex eigenvalue with multiplicity q.

– if $q = 1$, $B_j = C_2$

– T consists of columns that are real parts and imaginary parts of linearly independent solutions of $(A - \lambda I)^m$, where $m =$ multiplicity of eigenvalue λ. More precisely, columns of T can be calculated as follows

  * If λ is a real real eigenvalue with multiplicity m, then
    · find a basis of $N(A - \lambda I)$, say $V_1, V_2, ..., V_p$.
    · if the dim $N(A - \lambda I) < m$, then solve
    $$(A - \lambda I)^{m-1} W_j = V_j$$
    · all V_j, W_j, and $(A - \lambda I)^q W_j$, for $q = 1, 2, ..., m - 2$. Make sure they are linearly independent.
  * If λ is a complex eigenvalues.
    · Then find complex eigenvectors V and solutions of
    $$(A - \lambda I)^{m-1} X = V$$
    · real/imaginary parts of $V, W, (A - \lambda I)^q W, q = 1, 2, ..., m-2$
  * These vectors must be assembled according to the corresponding eigenvalues in B and the order of power (power q in $(A - \lambda I)^q W$)

Example 3 Find canonical forms for

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

Sol: eigenvalue $\lambda = 2$. Solve $(A-2)V = 0$. Since $\dim Nul\,(A-2I) = 1$, we have only one eigenvector in

$$V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

So we proceed to solve $(A-2I)^2 X = V_1$, or

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

There are two independent solutions

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

we choose the first one since the second one is a linear combination the the first one and V_1 :

$$V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Let

$$V_2 = (A-2)V_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

So

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad T = (V_1 V_2 V_3) = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

Example 4 Find canonical forms for

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Sol: $\det(A-\lambda) = \left(\lambda^2+1\right)^2 = 0$, $\lambda = \pm i$. Solve for $\lambda = i$,

$$(A-\lambda)X = 0 \rightarrow \quad X_1 = \begin{pmatrix} 1 \\ 1-i \\ 0 \\ 0 \end{pmatrix}, \; V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; V_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$(A-\lambda)^2 X = X_1 \rightarrow \quad X_2 = \begin{pmatrix} 0 \\ 0 \\ 1-i \\ 1 \end{pmatrix}, \; V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, V_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

So

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example 5 Find canonical forms for

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

Sol: $\det(A-\lambda) = (2-\lambda)^2\left((2-\lambda)^2+1\right) = 0,\ \lambda_1 = 2, m_1 = 2, \lambda_2 = 2 \pm i.$
For $\lambda_1 = 2$,

$$(A-2I)V_1 = 0 \rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A-2I)\,V_2 = V_1 \to V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2+i$,

$$(A-\lambda_2 I)\,X = 0 \to X = \begin{pmatrix} 0 \\ -i \\ 0 \\ 1 \end{pmatrix} \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; V_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

So one can choose

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

or

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad T = (V_3, V_4, V_1, V_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Genericity: The set of all $n \times n$ matrices with distinct eigenvalues is open and dense set in $L(R^n)$, the space of all $n \times n$. matrices
- Homework: 3, 5gh, 6, 10