Chapter 4 Classification of Planar Linear Systems

$$x' = ax + by$$
$$y' = cx + dy$$

Matrix form:

$$X' = AX$$
$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The characteristic polynomial for eigenvalues is

$$\det(A - \lambda I) = 0$$
$$\lambda^2 - (a+b)\lambda + ad - bc = 0$$

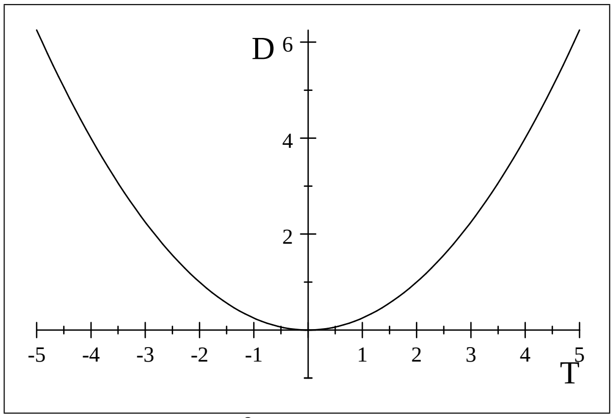
- Trace: $T = Tr(A) = a + b$
- Determinant: $D = \det(A)\, ad - bc$
- The characteristic equation is

$$\lambda^2 - T\lambda + D = 0$$

- Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \ \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$
$$T = \lambda_1 + \lambda_2, \quad D = \lambda_1 \lambda_2$$

- The Trace - Determinant Plane



$$T^2 - 4D = 0$$

- real distinct eigenvalues: $T^2 - 4D > 0$ (below the parabola)
    1. $D < 0 : \lambda_1 < 0 < \lambda_2$ saddle
    2. $D > 0$ and $T > 0 :\ 0 < \lambda_1 < \lambda_2$ source
    3. $D > 0$ and $T < 0 :\ \lambda_1 < \lambda_2 < 0$ sink
- repeated eigenvalues: $T^2 - 4D = 0$
    1. $T < 0 :\ \lambda_1 < 0$ sink
    2. $T > 0 :\ \lambda_1 > 0$ source
- complex eigenvalues: $T^2 - 4D < 0$ (above the parabola)
    1. spiral sink: $T < 0$ (left half of the plane)
    2. spiral source: $T > 0$ (right half of the plane)
    3. Center: $T = 0$ (D - axis)

• The above Trace - Determinant diagram can also be used as a bifurcation diagram.

• Conjugacy of two systems

- Two linear system $X' = AX$ and $Y' = BY$
- $\phi^A(t, X_0)$ is the flow of $X' = AX$,i.e., $X(t) = \phi^A(t, X_0)$ is the solution with initial data $X(0) = X_0$
- $\phi^B(t, Y_0)$ is the flow of $Y' = BY$
- $h : R^2 \to R^2$ is a homeomorphism (1-1, onto, h and $h^{-1}$are continuous)
- Definition: These two system is called conjugate if

$$\phi^B(t, h(X_0)) = h\left(\phi^A(t, X_0)\right)$$

- i.e., if $X(t)$ is a solution of $X' = AX$, then $h(X(t))$ is a solution of $Y' = BY$.
- h is called a conjugacy. h maps a solution for A to a solution for B.

• Example 1: For linear transformation T, it is also a conjugacy that maps a solution of $X' = AX$ to a solution of $Y' = (T^{-1}AT)Y$

- Example 2: Consider

$$x' = \lambda_1 x \text{ and } y' = \lambda_2 y$$

  - their flows are, respectively,

$$\phi^i(t, x_0) = x_0 e^{\lambda_i t}, \quad i = 1, 2$$

  - if both $\lambda_1$and λ_2 have the same sign, then they are conjugate with the conjugacy

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1} & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} & \text{if } x < 0 \end{cases}$$

  - verify: for $x_0 > 0$,

$$\begin{aligned} h\left(\phi^1(t, x_0)\right) &= h\left(x_0 e^{\lambda_1 t}\right) = \left(x_0 e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} \\ &= (x_0)^{\lambda_2/\lambda_1} e^{\lambda_2 t} = \phi^2\left(t, (x_0)^{\lambda_2/\lambda_1}\right) = \phi^2(t, h(x_0)) \end{aligned}$$

- Definition: A matrix is called hyperbolic if none of its eigenvalues has real part 0.

- Theorem: Two hyperbolic linear system $X' = A_i X$ are conjugate iff they have the same number of eigenvalues with negative real part, i.e.,

  - Both have one positive and one negative eigenvalue
  - Both have complex eigenvalues with positive real part (including both eigenvalues are real)
  - Both have complex eigenvalues with negative real part (including both eigenvalues are real)

- The proof in the case of real eigenvalues is similar to Example 2 for canonical forms.

- From this Theorem, all systems of sink (source) or spiral sink (spiral source) are conjugate with each other.

- Homework for Chapter 4: 1, 3, 4, 5a