Chapter 3 Phase Portraits for Planar Linear Systems

We shall develop an approach to solve linear systems in higher dimension. As an example, we look at planar systems from a different angle.

- Phase portrait of general autonomous planar systems

$$X' = F(X) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

  is the direction fields in $xy - plane$,in which at each point $X = (x, y)$ we assign a vector $F(X)$ of equal length.

- Canonical forms

  - canonical form for matrices with two distinct real eigenvalues

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

  - canonical form for matrices with repeated real eigenvalues

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \ or \ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

  - canonical form for matrices with complex eigenvalues $\lambda = \alpha + i\beta$

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

- Theorem: For any 2×2 matrix, there is an invertible matrix T such that $T^{-1}AT$ is in a canonical form. Moreover,

  - If $T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $T = (V_1 \ V_2)$, where V_i is an eigenvector associated with λ_i.

- if $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, then $T = (V_1 \ V_2)$, where $V = V_1 + iV_2$ is an complex eigenvector associated with $\lambda = \alpha + i\beta$. Reason:Since $AV = \lambda V$, or

$$\begin{aligned} AV_1 + iAV_2 &= (\alpha + i\beta)(V_1 + iV_2) \\ &= \alpha V_1 - \beta V_2 + i(\beta V_1 + \alpha V_2) \end{aligned}$$

so

$$AV_1 = \alpha V_1 - \beta V_2, \quad AV = \beta V_1 + \alpha V_2$$

and thus

$$\begin{aligned} AT &= (AV_1, AV_2) \\ &= (\alpha V_1 - \beta V_2, \ \beta V_1 + \alpha V_2) = T \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \end{aligned}$$

- Linear transformation, diagonalization, and changes of coordinates
  - For any matrix $T = (V_1 \ V_2)$, we call $X = TW$ is a linear transformation from $W = \binom{u}{v}$ space to $X = \binom{x}{y}$ space
  - for any vectors basis vectors V_1 and V_2, we call $W = \binom{u}{v}$ coordinate with respect to the basis $\{V_1 \ , \ V_2\}$ of the vector $(uV_1 \ + vV_2)$
  - for instance, $X = \binom{x}{y}$ is the coordinate of $X$with respect to the standard basis $\left\{\binom{1}{0}, \binom{0}{1}\right\}$
  - Since $\binom{x}{y} = X = TW = (V_1 \ V_2)\binom{u}{v} = uV_1 \ + vV_2$, we can see that $W = \binom{u}{v}$ is actually coordinate of $X = \binom{x}{y}$ with respect to the basis $\{V_1 \ , \ V_2\}$
  - So a linear transformation $T = (V_1 \ V_2)$ is also called a change of coordinate: It changes the coordinate $W = \binom{u}{v}$ with respect to the basis $\{V_1 \ , \ V_2\}$ to the coordinate for the standard basis $X = \binom{x}{y}$.
  - Now
$$W' = T^{-1}X' = T^{-1}AX = \left(T^{-1}AT\right)W$$
  - So T maps a solution curve for $W' = (T^{-1}AT)W$ to a solution curve of $X' = AX$, and vice versa.

– In other words, if $W = \binom{u(t)}{v(t)}$ solves $W' = (T^{-1}AT)\,W$, then $X = TW$ solves $X' = AX$.

- According to the Theorem above, $T^{-1}AT$ has three basic canonical forms.

From these discussion, we can easily find solutions and phase portraits:

- Solving Planar linear systems $X' = AX$

  1. A has two distinct real eigenvalue $\lambda_1 < \lambda_2$. Then their associated eigenvector V_1 and V_2 are linearly independent. using the linear transformation $T = (V_1\ V_2)$, $T^{-1}AT$ is in the canonical form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

     – General solutions of this canonical system is

$$u = c_1 e^{\lambda_1 t},\ v = c_2 e^{\lambda_2 t}$$

     or in vector form

$$W = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

     – General solutions for the original system $X' = AX$ is then

$$X = TW = (V_1\ V_2)\,W = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

  2. A has a pair of complex eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$. Let $V = V_1 + iV_2$ be a complex eigenvector associated with λ (both V_1 and V_2 are real vectors). Then $T^{-1}AT$ is in the canonical form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

     – General solutions of this canonical system is

$$W = c_1 W_1 + c_2 W_2 = e^{\alpha t} \begin{pmatrix} c_1 \cos \beta t + c_2 \sin \beta t \\ -c_1 \sin \beta t + c_2 \cos \beta t \end{pmatrix}$$

where

$$
\begin{aligned}
W_1 &= e^{\alpha t}\cos\beta t\begin{pmatrix}1\\0\end{pmatrix} - e^{\alpha t}\sin\beta t\begin{pmatrix}0\\1\end{pmatrix} = e^{\alpha t}\begin{pmatrix}\cos\beta t\\-\sin\beta t\end{pmatrix}\\
W_2 &= e^{\alpha t}\sin\beta t\begin{pmatrix}1\\0\end{pmatrix} + e^{\alpha t}\cos\beta t\begin{pmatrix}0\\1\end{pmatrix} = e^{\alpha t}\begin{pmatrix}\sin\beta t\\\cos\beta t\end{pmatrix}
\end{aligned}
$$

– General solutions of the original system is

$$X = TW = c_1TW_1 + c_2TW_2$$

where

$$
\begin{aligned}
X_1 &= TW_1 = e^{\alpha t}\left(\cos\beta t V_1 - \sin\beta t V_2\right)\\
X_2 &= TW_2 = e^{\alpha t}\left(\cos\beta t V_2 + \sin\beta t V_1\right)
\end{aligned}
$$

3. A has a repeated eigenvalue $\lambda_1 = \lambda_2$ with an eigenvector V_1.Then

$$
\begin{aligned}
X &= c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_1 t}\left(V_2 + tV_1\right)\\
&= e^{\lambda_1 t}\left[c_1V_1 + c_2V_2 + c_2tV_1\right]
\end{aligned}
$$

where V_2 is a solution of $\left(A - \lambda_1 I\right)V_2 = V_1$

- Phase Portraits of linear systems
- In the following discussion, we may assume canonical forms, i.e., $V_1 = \binom{1}{0},\ V_2 = \binom{0}{1}$.

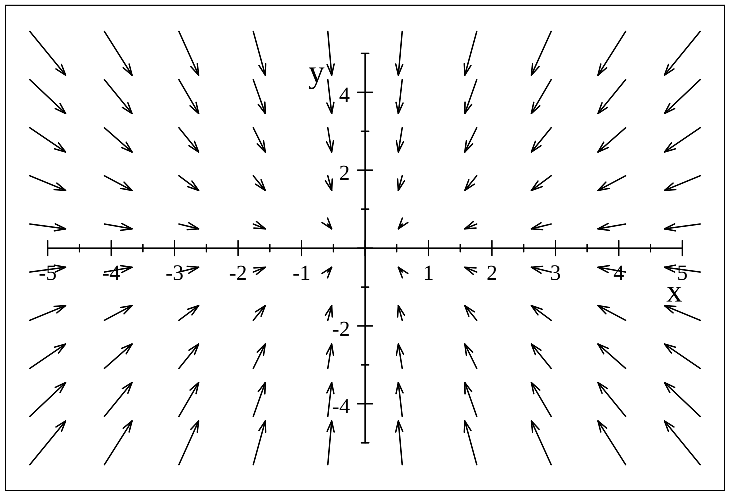
1. A has two distinct real eigenvalue $\lambda_1 < \lambda_2$ with associated eigenvectors V_1 and V_2.Then

$$
\begin{aligned}
X &= c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2\\
&= e^{\lambda_2 t}\left(c_1e^{(\lambda_1-\lambda_2)t}V_1 + c_2V_2\right) \to c_2e^{\lambda_2 t}V_2 \text{ as } t\to\infty
\end{aligned}
$$

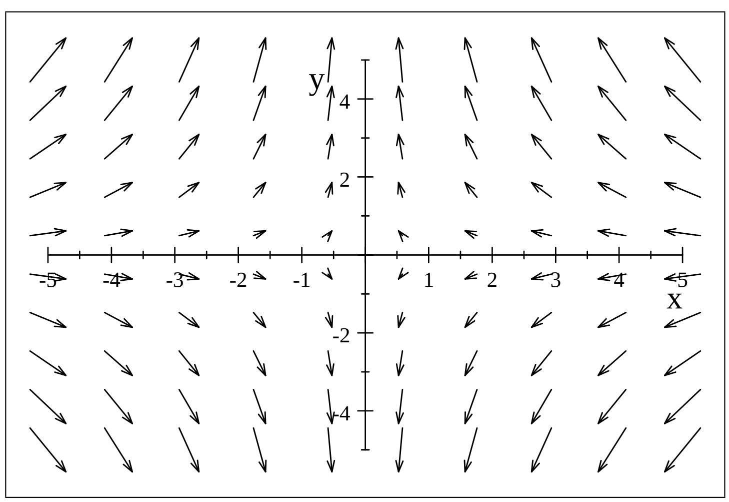
$$
\begin{aligned}
X &= c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2\\
&= e^{\lambda_1 t}\left(c_1V_1 + c_2e^{(\lambda_2-\lambda_1)t}V_2\right) \to c_1e^{\lambda_1 t}V_2 \text{ as } t\to-\infty
\end{aligned}
$$

So asymptotically, it behaves as

(a) $\lambda_1 < \lambda_2 < 0$ (sink)
As $t \to \infty$, solutions follows the direction of "large" V_2 approaches to zero.
As $t \to -\infty$, solutions follows the direction of "small" V_1 approaches to zero.

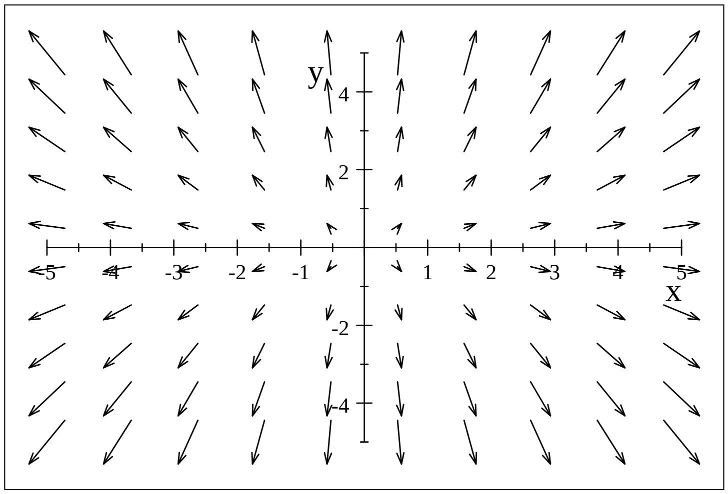


(b) $\lambda_1 < 0 < \lambda_2$ (saddle)
As $t \to \infty$, solutions follows the direction of "large" V_2 approaches to ∞.
As $t \to -\infty$, solutions follows the direction of "small" V_1 approaches to zero.



(c) $0 < \lambda_1 < \lambda_2$ (source)
As $t \to \infty$, solutions follows the direction of "large" V_2 ap-

proaches to ∞.
As $t \to -\infty$, solutions follows the direction of "small" V_1 approaches to ∞

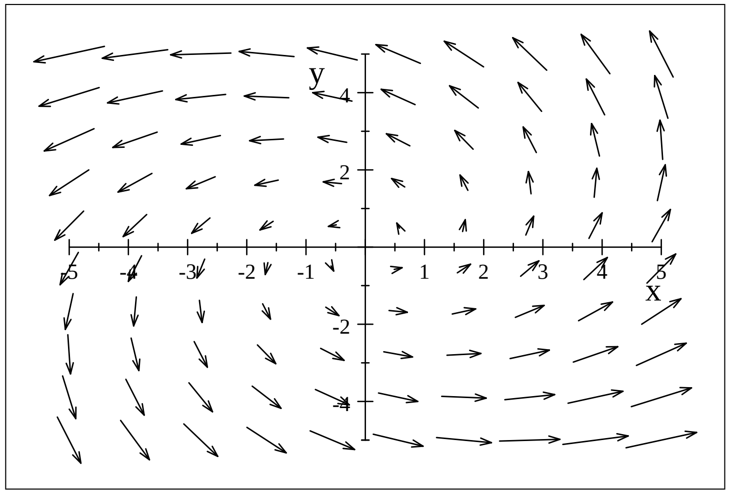


(d) one eigenvalue is zero (see homework #10,11)
   - $\lambda_1 < \lambda_2 = 0$ (sink)
   - $0 = \lambda_1 < \lambda_2$ (source)

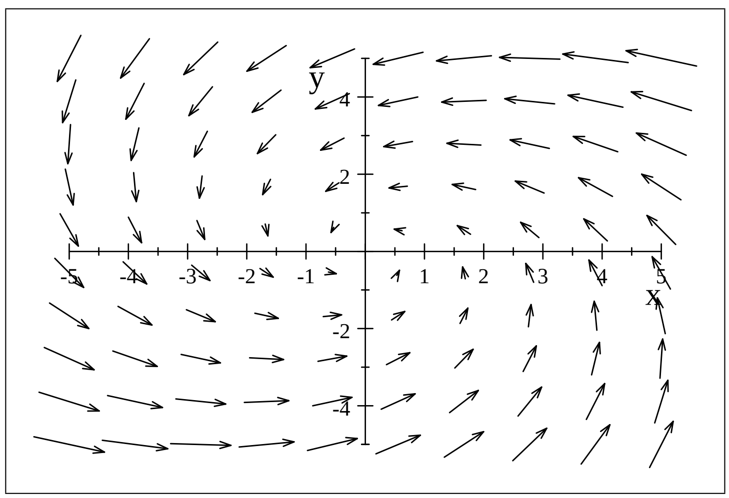
2. A has a complex eigenvalue $\lambda = \alpha + i\beta$ with eigenvector $V = V_1 + iV_2$.Then

$$\begin{aligned} X &= e^{\alpha t}\left[c_1\left(\cos\beta t V_1 - \sin\beta t V_2\right) + c_2\left(\cos\beta t V_2 + \sin\beta t V_1\right)\right] \\ &= e^{\alpha t}\left[c_1 \cos\beta t V_1 - c_1 \sin\beta t V_2 + c_2 \cos\beta t V_2 + c_2 \sin\beta t V_1\right] \\ &= e^{\alpha t}\left[\left(c_1 \cos\beta t + c_2 \sin\beta t\right) V_1 + \left(c_2 \cos\beta t - c_1 \sin\beta t\right) V_2\right] \end{aligned}$$

(a) $\alpha > 0$ (spiral source)
solutions $X(t) \to \infty$ as $t \to \infty$, $X(t) \to 0$ as $t \to -\infty$, in a spiral fashion around the origin.



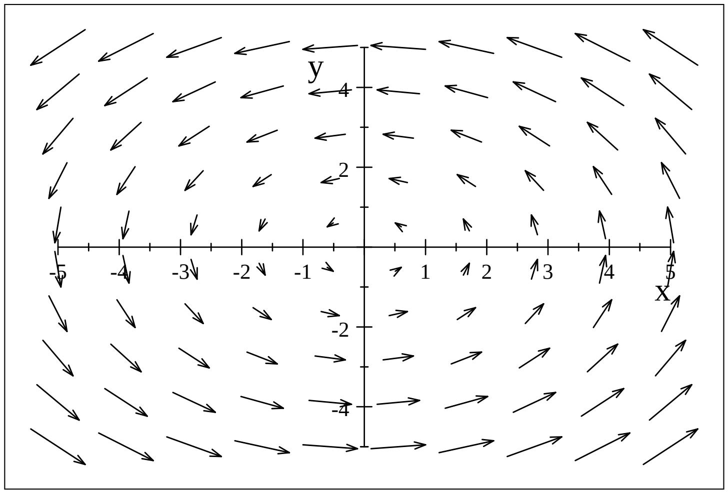
(b) $\alpha < 0$ (spiral sink)
solutions $X(t) \to 0$ as $t \to \infty$, $X(t) \to \infty$ as $t \to -\infty$, in a spiral fashion



(c) $\alpha = 0$ (center)

$$X = (c_1 \cos \beta t + c_2 \sin \beta t) V_1 + (c_2 \cos \beta t - c_1 \sin \beta t) V_2$$

is periodic with frequency β and period $T = 2\pi/\beta$.



- Homework for Chapter 3: 2(ii)(iii), 4, 5,10, 11