Chapter 2 Planar Linear Systems

$$x' = ax + by$$
$$y' = cx + dy$$

Matrix form:

$$X' = AX$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

There are four parameters: a, b, c, d.

- Question: How these 4 parameters determine solutions of systems?

Basic facts we learned before:

1. Equilibrium solutions:
   - If $\det(A) \neq 0$,then $X = 0$ is the only equilibrium solution.
   - If $\det(A) = 0$, then solutions of $AX = 0$ is a straight line.
2. Linearity Principle: If $X_1(t)$ and $X_2(t)$ are two solutions, then so are their linear combinations $X = \alpha X_1 + \beta X_2$for any constants α and β.
3. If λ is an eigenvalue of A and V is an eigenvector associated λ (i.e., $AV = \lambda V$), then $X = e^{\lambda t}V$ satisfies the linear system.
4. When λ is a real eigenvalue, then $X = e^{\lambda t}V$ is a real solution.
   - In the case there are two distinct real eigenvalues λ_1 and λ_2 with associated eigenvectors V_1 and V_2, respectively, then the general solution is $X = c_1 e^{\lambda_1 t}V_1 + c_2 e^{\lambda_2 t}V_2$
   - $\lambda_1 < \lambda_2 < 0$ sink, stable
   - $\lambda_1 < 0 < \lambda_2$ saddle, unstable
   - $0 < \lambda_1 < \lambda_2$ source, unstable
   - As $t \to \infty$, solutions follows V_2; as $t \to -\infty$, solutions follows V_1
   - Phase portraits.

- Example: $a = b = d = 1$, $c = 4$

5. When λ is a complex eigenvalue, then $X = e^{\lambda t} V$ is a complex solution, which is not a solution of the system. However, its real part and imaginary part are two linearly independent solutions.

   - Let $\lambda = \alpha \pm \beta i$. If $\alpha > 0$, spiral source. If $\alpha < 0$, spiral sink. If $a = 0$ center.
   - Example: $a = 6, b = -1, c = 5, d = 4$

6. If λ is a repeated eigenvalue of A, then $X = e^{\lambda t}\left(V_1 + tV\right)$ is a also a solution, where
$$\left(A - \lambda I\right) V_1 = V.$$
   This is because
$$\begin{aligned} X' &= \lambda e^{\lambda t} V_1 + e^{\lambda t} V + \lambda t e^{\lambda t} V = e^{\lambda t}\left(\lambda V_1 + V + \lambda t V\right) \\ &= e^{\lambda t}\left(\lambda V_1 + V + \lambda t V\right) \\ AX &= e^{\lambda t}\left(A V_1 + t A V\right) = e^{\lambda t}\left(A V_1 + \lambda t V\right). \end{aligned}$$
   $AV_1 = \lambda V_1 + V$ leads to $X' = AX$.

7. Second-order ODE: $x'' + bx' + kx$can be converted into a system of first-order equations.

- Homework: #3, 6.