Chapter 10 Closed orbits and Limit Sets

Consider nonlinear dynamical system

\[ X' = F(X) \]

with flow \( \phi(t, X) \), i.e., \( X(t) = \phi(t, X_0) \) is the solution with the initial value \( X(0) = X_0 \).

In this chapter, we study the limit behavior of flow \( \phi(t, X) \).

- Recall Liapunov Stability Theorem:
  - A function defined in a neighborhood \( O \) of \( X_0 \) satisfying
    \[ a \; L(X_0) = 0, \; L(X) > 0 \; \text{for} \; X \neq X_0, \; X \in O \]
    \[ b \; \nabla L(X) \cdot F(X) < 0 \; \text{for} \; X \in O \setminus \{X_0\} \]
  - **Liapunov Stability Theorem**: Let \( X_0 \) be an equilibrium for \( X = F(X) \). The equilibrium is asymptotically stable if there exists a smooth Liapunov function.
    - Furthermore, for any \( X \in O \), \( \phi(t, X) \to X_0 \) as \( t \to \infty \).

- Definition: Basin of Attraction of an equilibrium \( X_0 \) is \( \{X : \phi(t, X) \to X_0 \; \text{as} \; t \to \infty\} \)
  - In **Liapunov Stability Theorem**, the set \( O \) lies inside of the Basin of Attraction of \( X_0 \).

Example. Consider, for parameter \( \varepsilon \),

\[
\begin{align*}
  x' &= (-x + 2y)(z + 1) \\
  y' &= (-x - y)(z + 1) \\
  z' &= -z^3
\end{align*}
\]

Consider a Liapunov function in the form:

\[ L(x, y, z) = ax^2 + by^2 + cz^2 \]

We see that

\[
\nabla L \cdot F = 2(ax, by, cz) \cdot F \\
= 2ax (-x + 2y)(z + 1) + 2by (-x - y)(z + 1) + 2cz (-z^3) \\
= -2(ax^2 + by^2)(z + 1) + (2a - b)yx (z + 1) - 2cz^4
\]
So if we choose $a = 1, b = 2, c = 1$. Then for all $X = (x, y, z)$

$$\nabla L \cdot F (X) = -2 (ax^2 + by^2) (z + 1) - 2cz^4 < 0$$

if $z > -1$

Hence, the entire half plane $z > -1$ lies in the Basin of Attraction of $O$.

- **Remark**: By Choosing appropriate the Liapunov function, we may find larger area inside Basin of Attraction, or even describe it precisely.

- Closed orbit: Periodic solution. If $\phi (T, X_0) = X_0$, then this solution $X (t) = \phi (t, X_0)$ is periodic with period $T$

- $\omega$—limit point of $X$: a point $Y$ is called a $\omega$—limit if there is a sequence $t_n \to \infty$ as $n \to \infty$ such that

$$Y = \lim_{n \to \infty} \phi (t_n, X)$$

- $\omega (X)$: $\omega$—limit set of $X = \text{Set of all possible } \omega$—limit points of $X$

- We also call $\omega$—limit set of $X$ "$\omega$—limit set of the function $\phi (T, X)$". In particular,

$$\omega$—limit set of $f (x) = \left\{ y : y = \lim_{x_n \to \infty} f (x_n) \right\}$$

- Illustration about sequential limit

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\[ y = \sin x : \omega$—limit = $[-1, 1] \]
- $\alpha-$ limit point of $X$: a point $Y$ is called an $\alpha-$ limit if there is a sequence $t_n \to -\infty$ as $n \to \infty$ such that
  \[ Y = \lim_{n \to -\infty} \phi(t_n, X) \]

- $\alpha(X)$: $\alpha-$ limit set = Set of all possible $\alpha-$ limit points of $X$

- Limit set means $\omega(X)$ or $\alpha(X)$

- Invariant set: A set $G$ is called invariant if for any $X \in G$, $\phi(t, X) \in G$ for all $t$.

- Positively Invariant set: A set $G$ is called Positively invariant if for any $X \in G$, $\phi(t, X) \in G$ for all $t > 0$.

- Negatively Invariant set: A set $G$ is called Negatively invariant if for any $X \in G$, $\phi(t, X) \in G$ for all $t < 0$.

- Any limit set is invariant:
  
  - For instance, suppose $Y \in \omega(X)$. If $\phi(t_n, X) \to Y$, then $\phi(t + t_n, X) = \phi(t, \phi(t_n, X)) \to \phi(t, Y)$. So $\phi(t, Y) \in \omega(X)$

Example 1: Consider the system

\[
\begin{align*}
x' &= \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \\
y' &= x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2y)
\end{align*}
\]

Introducing the polar coordinate system $x = r \cos \theta, y = r \sin \theta$

\[
\begin{align*}
x' &= r' \cos \theta - r \theta' \sin \theta \\
y' &= r' \sin \theta + r \theta' \cos \theta
\end{align*}
\]

So

\[
\begin{align*}
x' \cos \theta &= r' \cos^2 \theta - r \theta' \sin \theta \cos \theta \\
y' \sin \theta &= r' \sin^2 \theta + r \theta' \cos \theta \sin \theta
\end{align*}
\]
Adding together, we see

\[ x' \cos \theta + y' \sin \theta = r' \cos^2 \theta + r' \sin^2 \theta = r' \]

So

\[ r' = x' \cos \theta + y' \sin \theta \]

\[ = \left( \frac{1}{2} x - y - \frac{1}{2} (x^3 + xy^2) \right) \cos \theta + \left( x + \frac{1}{2} y - \frac{1}{2} (y^3 + x^2 y) \right) \sin \theta \]

\[ = \left( \frac{1}{2} r \cos \theta - r \sin \theta - \frac{1}{2} r^3 \cos \theta \right) \cos \theta + \left( r \cos \theta + \frac{1}{2} r \sin \theta - \frac{1}{2} r^3 \sin \theta \right) \sin \theta \]

\[ = \left( \frac{1}{2} r \cos^2 \theta - r \sin \theta \cos \theta - \frac{1}{2} r^3 \cos \theta \right) + \left( r \cos \theta \sin \theta + \frac{1}{2} r \sin^2 \theta - \frac{1}{2} r^3 \sin^2 \theta \right) \]

\[ = \frac{1}{2} (r - r^3) \]

Next, from

\[ x' = r' \cos \theta - r \theta' \sin \theta \]

we see

\[ \left( \frac{1}{2} x - y - \frac{1}{2} (x^3 + xy^2) \right) = \frac{1}{2} (r - r^3) \cos \theta - r \theta' \sin \theta \]

\[ \left( \frac{1}{2} r \cos \theta - r \sin \theta - \frac{1}{2} r^3 \cos \theta \right) = \frac{1}{2} r \cos \theta \sin \theta - \frac{1}{2} r^3 \sin \theta - r \theta' \sin \theta \]

\[ -r \sin \theta = -r \theta' \sin \theta \]

\[ \theta' = 1 \]

We thus arrive at

\[ r' = \frac{1}{2} (r - r^3) \]

\[ \theta' = 1 \]

\( r = 1, \theta = t \) is an solution, which is the unit circle under the polar coordinate system. We call it a closed orbit.

Consider any solution \((r (t), \theta (t))\) with the initial value \((r_0, \theta_0) : (r (0), \theta (0)) = (r_0, \theta_0)\). If \(0 < r_0 < 1\), then

\[ r' (0) = \frac{1}{2} r (1 - r^2) = \frac{1}{2} r_0 (1 - r_0^2) > 0 \]
So $r'(t)$ will stay positive for all $t > 0$. Hence, $r(t)$ will increases towards $r = 1$ as $t \to \infty$. Otherwise, if initially $r_0 > 1$, $r'(0) < 0$, and it will decreases to $r = 1$. So $\omega$– limit set of any nonzero solution is this orbit $r = 1$.

- Properties of limit sets:
  - Any limit set is closed and and invariant
  - If $X$ and $Z$ lie on the same solution, then they have the same limit sets: $\omega(X) = \omega(Z)$, $\alpha(X) = \alpha(Z)$
  - If $G$ is closed, positively invariant set, and $Z \in G$, then $\omega(Z) \subset G$.
  - If $G$ is closed, negatively invariant set, and $Z \in G$, then $\alpha(Z) \subset G$.

- Local Sections
  - Transverse line $l(X_0)$ at an non-equilibrium point $X_0$ is a line passing $X_0$ and is perpendicular to $F(X_0)$ (assume $F(X_0) \neq 0$)
  - Transverse line may be parametrized using vector for of line equation:
    \[ X = h(u) = X_0 + uV_0, \quad V_0 \text{ is a unit vector in the direction of } l(X_0), \quad V_0 \cdot F(X_0) = 0 \]
  - Local section - A line segment $S$ on $l(X_0)$ containing $X_0$ and is not tangent to the vector field.
  - At each point on $S$, there is a solution curve passing through it.
  - One may "straighten" the local section and flows nearby to form "flow box"

- Flow Boxes
  - Define mapping
    \[ \psi(s, u) = \psi(s, h(u)) \]
  - This mapping maps a rectangular box $[-\varepsilon, \varepsilon] \times [-\delta, \delta]$ to a neighborhood of local section $S$.
  - The image $\psi([-\varepsilon, \varepsilon] \times [-\delta, \delta])$ is called a flow box
  - $\psi(s, u)$ is a local conjugacy between the constant vector field and nonlinear vector field $F(X)$
• Closed orbit
  - A periodic solution is called a closed orbit
  - A closed orbit \( \gamma \) is called Stable if there exists a neighborhood \( G \) of \( \gamma \) and a sub-neighborhood \( G_1 \subset G \) such that any solution initiated from \( G_1 \) will remain in \( G \).
  - A closed orbit \( \gamma \) is called Asymptotically Stable if it is stable and \( \omega \)- limit set \( \omega (X_0) \subset \gamma \), for any \( X_0 \) near \( \gamma \). In other words, all solution will tend to \( \gamma \).

• The Poincaré Map
  - Definition: Given a closed orbit \( \gamma \) and \( X_0 \in \gamma \). Let \( S = l(X_0) \) be a local transverse section at \( X_0 \). For any \( X \in S \) near \( X_0 \), the flow \( \phi (t, X) \) may hit \( S \) at same time \( s = t(X) \). The map \( X \mapsto \phi (s, X) \) is called the Poincaré Map
    \[
    P(X) = \phi (s, X), \text{ } s \text{ is the first time the flow } \phi (t, X) \text{ hit } S
    \]
  - Note that \( P (X_0) = X_0 \).
  - In \( \mathbb{R}^2 \), a transverse section \( S \) may be represented by \( h(u) = X_0 + uV_0 \). So when restricted on \( S \), The Poincaré Map induces a mapping \( P_1 : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) as
    \[
    P_1 (u) = v, \text{ where } v \text{ is the first time } \phi (t, h(u)) \text{ hits } S \text{ at } h(v): P_1 (h(u)) = h(v), \text{ } P_1 (0) = 0.
    \]
  - If \( |P_1' (0)| < 1 \), then \( P_1 (u) = au + O(u^2) \), \( a = P_1' (0) \), \( |a| < 1 \). This means that after the first passage across \( S \), for any point \( X \) near \( X_0 \), its distance to \( X_0 \) decreases. Continuing, we see that \( \phi (t, X) \rightarrow X_0 \). Therefore, \( \gamma \) is asymptotically stable.

• Theorem: Let \( \gamma \) be a closed orbit and \( X_0 \in \gamma \). Let \( P \) be a Poincaré Map defined in a neighborhood of \( X_0 \). If \( |P' (X_0)| < 1 \), then \( \gamma \) is asymptotically stable.
Example: Consider system in the polar coordinate system

\[ r' = r(1 - r) \]
\[ \theta' = 1 \]

We know that \( r = 1 \) is the closed orbit. The corresponding solution in \( xy-plane \) is \((\cos t, \sin t)\). Let \( X_0 = (1, 0) \in \gamma \). The positive x-axis is a local transverse section. For any \( X = (x, 0), x > 0, \)

\[ P(x) = \phi(X, 2\pi). \]

In fact, we can solve \( r \) by separation of variables:

\[
\int \left( \frac{1}{r} + \frac{1}{1-r} \right) dr = \int \frac{dr}{r(1-r)} = \int d\theta
\]

\[
\ln r - \ln (1-r) = \theta + C
\]

or

\[
\frac{r}{1-r} = Ce^\theta = Ce^t \quad \text{ (since } \theta' = 1) \]

Since initially

\[ r(0) = \sqrt{x^2 + y^2} = x \]

we have

\[ C = \frac{x}{1-x} \]

So

\[ r = \frac{Ce^t}{1+Ce^t} = \frac{x}{1-x} \cdot e^t = \frac{xe^t}{1 - xe^t} = \frac{xe^t}{1 - x + xe^t} \]

Therefore,

\[ P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \]

\[ P'(x) = \frac{d}{dx} \left( \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \right) = \frac{e^{2\pi}}{(xe^{2\pi} - x + 1)^2} \]

In particular at \( X_0 = (1, 0), P'(1) = 1/e^{2\pi} < 1 \). So the orbit is asymptotically stable.
• **Poincaré-Bendixson Theorem**: Suppose that $\Omega$ is a closed and bounded limit set of a planar dynamical system. If $\Omega$ contains no equilibrium solution, then $\Omega$ is a closed orbit.

• Applications:
  
  - Limit cycle - a closed orbit $\gamma$ that is included in a limit set.
    * $\omega-$ limit cycle $\gamma$ if $\gamma \subset \omega(X)$ for some $X$
    * $\alpha-$ limit cycle $\gamma$ if $\gamma \subset \alpha(X)$ for some $X$
  
  - Limit cycle theorem: Let $\gamma$ be a $\omega-$ limit cycle, and $\gamma = \omega(X)$ for some $X \notin \gamma$. Then the set $\{Y : \omega(Y) = \gamma\} \setminus \gamma$ is an open set. In other words, for $Y$ near $X$, $\gamma = \omega(Y)$.
  
  - Any closed and bounded positively (or negatively) invariant set contains either a limit cycle or an equilibrium.
  
  - Let $\gamma$ be a closed orbit and $G$ be the open region inside and bounded by $\gamma$. Then $G$ contains either an equilibrium.

• Homework: 1abc, 2, 5, 6 (for 6c, see the example in at the bottom of page 207)