Chapter 10 Closed orbits and Limit Sets

Consider nonlinear dynamical system

$$X' = F(X)$$

with flow $\phi(t, X)$, i.e., $X(t) = \phi(t, X_0)$ is the solution with the initial value $X(0) = X_0$

In this chapter, we study the limit behavior of flow $\phi(t, X)$.

- Recall Liapunov Stability Theorem:
  - A function defined in a neighborhood O of X_0 satisfying

    a $L(X_0) = 0,\ L(X) > 0$ for $X \neq X_0,\ X \in O$

    b $\nabla L(X) \cdot F(X) < 0$ for $X \in O \setminus \{X_0\}$
  - **Liapunov Stability Theorem**: Let X_0 be an equilibrium for $X = F(X)$. The equilibrium is asymptotically stable if there exists a smooth Liapunov function.
  - Furthermore, for any $X \in O,\ \phi(t, X) \to X_0$ as $t \to \infty$.
- Definition: Basin of Attraction of an equilibrium X_0 is $\{X : \phi(t, X) \to X_0 \text{ as } t \to \infty\}$
  - In **Liapunov Stability Theorem, the set** O lies inside of the Basin of Attraction of X_0.

Example . Consider, for parameter ε,

$$\begin{aligned} x' &= (-x + 2y)(z + 1) \\ y' &= (-x - y)(z + 1) \\ z' &= -z^3 \end{aligned}$$

Consider a Liapunov function in the form:

$$L(x, y, z) = ax^2 + by^2 + cz^2$$

We see that

$$\begin{aligned} \nabla L \cdot F &= 2(ax, by, cz) \cdot F \\ &= 2ax(-x + 2y)(z + 1) + 2by(-x - y)(z + 1) + 2cz\left(-z^3\right) \\ &= -2\left(ax^2 + by^2\right)(z + 1) + (2a - b)\,yx(z + 1) - 2cz^4 \end{aligned}$$

So if we choose $a=1, b=2, c=1$. Then for all $X=(x,y,z)$

$$\nabla L \cdot F(X) = -2\left(ax^2+by^2\right)(z+1) - 2cz^4 < 0$$
$$\text{if } z > -1$$

Hence, the entire half plane $z > -1$ lies in the Basin of Attraction of O.

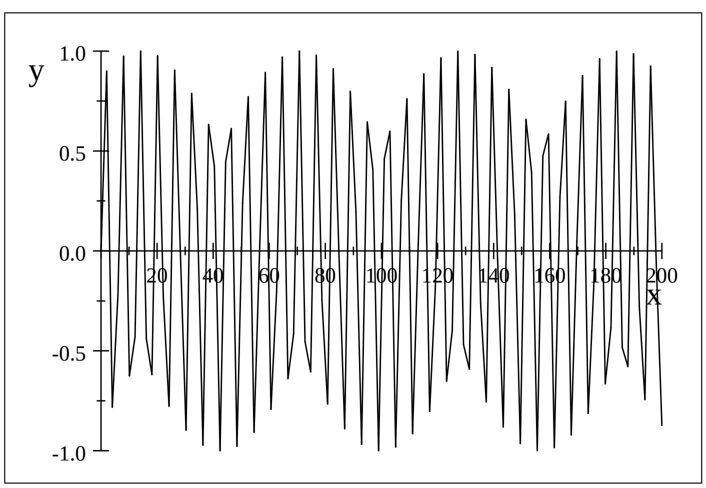
- **Remark**: By Choosing appropriate the Liapunov function, we may find larger area inside Basin of Attraction, or even describe it precisely.
- Closed orbit : Periodic solution. If $\phi(T, X_0) = X_0$, then this solution $X(t) = \phi(t, X_0)$ is periodic with period T
- $\omega-$ limit point of X : a point Y is called a $\omega-$ limit if there is a sequence $t_n \to \infty$ as $n \to \infty$ such that

$$Y = \lim_{n\to\infty} \phi(t_n, X)$$

- $\omega(X)$: $\omega-$ limit set of X = Set of all possible $\omega-$ limit points of X
- We also call $\omega-$ limit set of X "$\omega-$ limit set of the function $\phi(T,X)$". In particular,

$$\omega - \text{limit set of } f(x) = \left\{ y : y = \lim_{x_n\to\infty} f(x_n) \right\}$$

- Illustration about sequential limit



$y = \sin x : \omega\text{−limit} = [-1, 1]$

- $\alpha-$ limit point of X : a point Y is called a $\alpha-$ limit if there is a sequence $t_n \to -\infty$ as $n \to \infty$ such that

$$Y = \lim_{n \to -\infty} \phi(t_n, X)$$

- $\alpha(X)$: $\alpha-$ limit set = Set of all possible $\alpha-$ limit points of X

- Limit set means $\omega(X)$ or $\alpha(X)$

- Invariant set: A set G is called invariant if for any $X \in G,\ \phi(t, X) \in G$ for all t.

- Positively Invariant set: A set G is called Positively invariant if for any $X \in G,\ \phi(t, X) \in G$ for all $t > 0$.

- Negatively Invariant set: A set G is called Negatively invariant if for any $X \in G,\ \phi(t, X) \in G$ for all $t < 0$.

- Any limit set is invariant:

  - For instance, suppose $Y \in \omega(X)$. If $\phi(t_n, X) \to Y$, then $\phi(t + t_n, X) = \phi(t, \phi(t_n, X)) \to \phi(t, Y)$. So $\phi(t, Y) \in \omega(X)$

Example 1: Consider the system

$$x' = \frac{1}{2}x - y - \frac{1}{2}\left(x^3 + xy^2\right)$$
$$y' = x + \frac{1}{2}y - \frac{1}{2}\left(y^3 + x^2y\right)$$

Introducing the polar coordinate system $x = r\cos\theta, y = r\sin\theta$

$$x' = r'\cos\theta - r\theta'\sin\theta$$
$$y' = r'\sin\theta + r\theta'\cos\theta$$

So

$$x'\cos\theta = r'\cos^2\theta - r\theta'\sin\theta\cos\theta$$
$$y'\sin\theta = r'\sin^2\theta + r\theta'\cos\theta\sin\theta$$

Adding together, we see

$$x' \cos\theta + y' \sin\theta = r' \cos^2\theta + r' \sin^2\theta = r'$$

So

$$\begin{aligned} r' &= x' \cos\theta + y' \sin\theta \\ &= \left(\frac{1}{2}x - y - \frac{1}{2}\left(x^3 + xy^2\right)\right)\cos\theta + \left(x + \frac{1}{2}y - \frac{1}{2}\left(y^3 + x^2 y\right)\right)\sin\theta \\ &= \left(\frac{1}{2}r\cos\theta - r\sin\theta - \frac{1}{2}r^3\cos\theta\right)\cos\theta + \left(r\cos\theta + \frac{1}{2}r\sin\theta - \frac{1}{2}r^3\sin\theta\right)\sin\theta \\ &= \left(\frac{1}{2}r\cos^2\theta - r\sin\theta\cos\theta - \frac{1}{2}r^3\cos^2\theta\right) + \left(r\cos\theta\sin\theta + \frac{1}{2}r\sin^2\theta - \frac{1}{2}r^3\sin^2\theta\right) \\ &= \frac{1}{2}\left(r - r^3\right) \end{aligned}$$

Next, from

$$x' = r' \cos\theta - r\theta' \sin\theta$$

we see

$$\begin{aligned} \left(\frac{1}{2}x - y - \frac{1}{2}\left(x^3 + xy^2\right)\right) &= \frac{1}{2}\left(r - r^3\right)\cos\theta - r\theta'\sin\theta \\ \left(\frac{1}{2}r\cos\theta - r\sin\theta - \frac{1}{2}r^3\cos\theta\right) &= \frac{1}{2}r\cos\theta - \frac{1}{2}r^3\cos\theta - r\theta'\sin\theta \\ -r\sin\theta &= -r\theta'\sin\theta \\ \theta' &= 1 \end{aligned}$$

We thus arrive at

$$\begin{aligned} r' &= \frac{1}{2}\left(r - r^3\right) \\ \theta' &= 1 \end{aligned}$$

$r = 1, \theta = t$ is an solution, which is the unit circle under the polar coordinate system. We call it a closed orbit.

Consider any solution $(r(t), \theta(t))$with the initial value $(r_0, \theta_0) : (r(0), \theta(0)) = (r_0, \theta_0)$. If $0 < r_0 < 1$, then

$$r'(0) = \frac{1}{2}r\left(1 - r^2\right) = \frac{1}{2}r_0\left(1 - r_0^2\right) > 0$$

So $r'(t)$ will stay positive for all $t > 0$. Hence, $r(t)$ will increases towards $r = 1$ as $t \to \infty$. Otherwise, if initially $r_0 > 1$, $r'(0) < 0$, and it will decreases to $r = 1$. So $\omega-$ limit set of any nonzero solution is this orbit $r = 1$.

- Properties of limit sets:
  - Any limit set is closed and and invariant
  - If X and Z lie on the same solution, then they have the same limit sets: $\omega(X) = \omega(Z)\,,\ \alpha(X) = \alpha(Z)$
  - If G is closed, positively invariant set, and $Z \in G$, then $\omega(Z) \subset G$.
  - If G is closed, negatively invariant set, and $Z \in G$, then $\alpha(Z) \subset G$.
- Local Sections
  - Transverse line $l(X_0)$ at an non-equilibrium point X_0 is a line passing X_0 and is perpendicular to $F(X_0)$ (assume $F(X_0) \neq \vec{0}$)
  - Transverse line may be parametrized using vector for of line equation:

    $$X = h(u) = X_0 + uV_0,\ V_0 \text{ is a unit vector in the direction of } l(X_0)\,,\ V_0 \cdot F(X_0) = 0$$

  - Local section - A line segment S on $l(X_0)$ containing X_0 and is not tangent to the vector field.
  - At each point on S, there is a solution curve passing through it.
  - One may "straighten" the local section and flows nearby to form "flow box"
- Flow Boxes
  - Define mapping
    $$\psi(s,u) = \phi(s, h(u))$$
  - This mapping maps a rectangular box $[-\varepsilon,\varepsilon] \times [-\delta,\delta]$ to a neighborhood of local section S.
  - The image $\psi([-\varepsilon,\varepsilon] \times [-\delta,\delta])$ is called a flow box
  - $\psi(s,u)$ is a local conjugacy between the constant vector field and nonlinear vector field $F(X)$

- Closed orbit
  - A periodic solution is called a closed orbit
  - A closed orbit γ is called Stable if there exists a neighborhood G of γ and a sub-neighborhood $G_1 \subset G$ such that any solution initiated from G_1 will remain in G.
  - A closed orbit γ is called Asymptotically Stable if it is stable and $\omega-$ limit set $\omega(X_o) \subset \gamma$, for any X_0 near γ. In other words, all solution will tend to γ.
- The Poincaré Map
  - Definition: Given a closed orbit γ and $X_0 \in \gamma$. Let $S = l(X_0)$ be a local transverse section at X_0. For any $X \in S$ near X_0, the flow $\phi(t, X)$ may hit S at same time $s = t(X)$. The map $X \longmapsto \phi(s, X)$ is called the Poincaré Map

    $$P(X) = \phi(s, X), \ s \text{ is the first time the flow } \phi(t, X) \text{ hit } S$$

  - Note that $P(X_0) = X_0$.
  - In R^2, a transverse section S may be represented by $h(u) = X_0 + uV_0$. So when restricted on S, The Poincaré Map induces a mapping $P_1 : R^1 \to R^1$ as

    $$P_1(u) = v, \text{ where } v \text{ is the first time } \phi(t, h(u)) \text{ hits } S \text{ at } h(v):$$

    $$P_1(h(u)) = h(v), \quad P_1(0) = 0.$$

  - If $|P_1'(0)| < 1$, then $P_1(u) = au + O(u^2)$, $a = P_1'(0)$, $|a| < 1$. This means that after the first passage across S, for any point X near X_0, its distance to X_0 decreases. Continuing, we see that $\phi(t, X) \to X_0$. Therefore, γ is asymptotically stable.
- Theorem: Let γ be a closed orbit and $X_0 \in \gamma$. Let P be a Poincaré Map defined in a neighborhood of X_0. If $|P'(X_0)| < 1$, then γ is asymptotically stable.

Example: Consider system in the polar coordinate system

$$r' = r(1-r)$$
$$\theta' = 1$$

We know that $r = 1$ is the closed orbit. The corresponding solution in $xy - plane$ is $(\cos t, \sin t)$. Let $X_0 = (1,0) \in \gamma$. The positive x-axis is a local transverse section. For any $X = (x, 0)$, $x > 0$,

$$P(x) = \phi(X, 2\pi).$$

In fact, we can solve r by separation of variables:

$$\int \left(\frac{1}{r} + \frac{1}{1-r}\right) dr = \int \frac{dr}{r(1-r)} = \int d\theta$$

$$\ln r - \ln(1-r) = \theta + C$$

or

$$\frac{r}{1-r} = Ce^{\theta} = Ce^{t} \quad (\text{since } \theta' = 1)$$

Since initially

$$r(0) = \sqrt{x^2 + y^2} = x$$

we have

$$C = \frac{x}{1-x}$$

So

$$r = \frac{Ce^t}{1 + Ce^t} = \frac{\frac{x}{1-x}e^t}{1 + \frac{x}{1-x}e^t} = \frac{xe^t}{1 - x + xe^t}$$

Therefore,

$$P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}}$$

$$P'(x) = \frac{d}{dx}\left(\frac{xe^{2\pi}}{1 - x + xe^{2\pi}}\right) = \frac{e^{2\pi}}{\left(xe^{2\pi} - x + 1\right)^2}$$

In particular at $X_0 = (1, 0)$, $P'(1) = 1/e^{2\pi} < 1$. So the orbit is asymptotically stable.

- **Poincaré-Bendixson Theorem**: Suppose that Ω is a closed and bounded limit set of a planar dynamical system. If Ω contains no equilibrium solution, then Ω is a closed orbit.

- Applications:

  - Limit cycle - a closed obit γ that is included in a limit set.
    - $\omega-$ limit cycle γ if $\gamma \subset \omega(X)$ for some X
    - $\alpha-$ limit cycle γ if $\gamma \subset \alpha(X)$ for some X
  - Limit cycle theorem: Let γ be a $\omega-$ limit cycle, and $\gamma = \omega(X)$ for some $X \notin \gamma$. Then the set $\{Y : \omega(Y) = \gamma\} \setminus \gamma$ is an open set. In other words, for Y near X, $\gamma = \omega(Y)$.
  - Any closed and bounded positively (or negatively) invariant set contains either a limit cycle or an equilibrium.
  - Let γ be a closed orbit and G be the open region inside and bounded by γ.Then G contains either an equilibrium.

- Homework: 1abc, 2, 5, 6 (for 6c, see the example in at the bottom of page 207)