MTH 4810/6810 Applied Mathematics I
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Chapter 1 First-Order Equations

We start with a brief review about basics of ODE:

$$x' = f(t, x)$$

where $f(t, x)$ is a given two variable function. Some examples:

1. Simple population model: Assuming that the rate of change of a population is proportional to its size. Then

   $$x' = ax$$

   $a(t) =$ size of population of a species at time t.

2. Logistic population model: In addition to simple growth assumption, we assume that there is a capacity limit N. Beyond this limit, the population will decrease:

   $$x' = ax\left(1 - \frac{x}{N}\right)$$

3. Logistic population model with harvesting (Exercise: find general solution):

   $$x' = ax\left(1 - \frac{x}{N}\right) - h$$

   - $h -$ rate of harvesting

- Review basic methods of analysing ODEs

- Analytical Method: Separation of Variables
  - Example 1: Solve Logistic model

$$x' = ax\left(1 - \frac{x}{N}\right)$$

- Geometric Method: Directional fields
  - Phase lines of autonomous systems $x' = f(x)$
  - equilibrium solutions: solutions of $f(x) = 0$
  - classification of equilibria: sink (stable), source, saddle
  - Example 2: $x' = ax$
    - Sol: General solution is $x = ke^{at}$
    - Equilibrium solution is $x = 0$.
    - If $a > 0$, unstable. If $a < 0$, $x = ke^{at} \to 0$ as $t \to \infty$, stable.
    - When a changes from > 0 to < 0, the behavior of solutions change.
    - So $a = 0$ is called a bifurfaction point.
  - Example 3 Re-consider logistic model. $x = 0, N$ are equilibria.
    - Draw phase line to determine $x = 0$ is a source and $x = N$ is a sink if $a > 0$.
    - Converse If $a < 0$.
    - So $a = 0$ is a bifurcation, since the behavior of solutions changes when a crosses $a = 0$.
- In general, one can easily determine sink/source
  - if $f'(x_0) < 0$, an equilibrium $x = x_0$ is a sink
  - if $f'(x_0) > 0$, an equilibrium $x = x_0$ is a source
  - But if $f'(x_0) = 0$, anything could happen: it could be a sink, or source, or saddle.
- Example 4 Discuss equilibria for $x' = x - x^3$

• Consider ODEs depending on a parameter a : $x' = f(a, x)$

- Question: How solutions change as the parameter a changes?

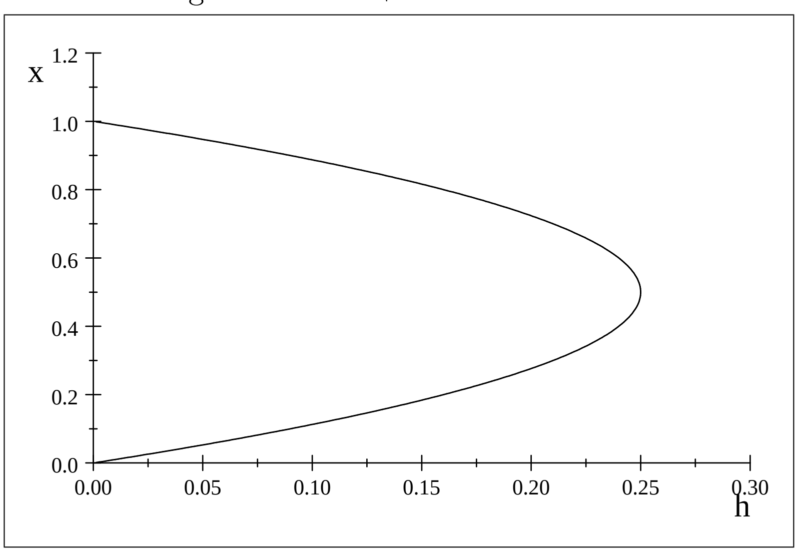
- Bifurcation diagrams for $x' = f(a,x)$
  - On $ax - plane$, we graph the equation $f(a,x) = 0$
  - The is the curve of equilibrium solutions and is called the bifurcation diagram
  - Let $x(a)$ be solutions.
  - Bifurcation occurs when solutions $x(a)$ change in certain way
  - For instance, number of solutions changes, types change (from sink to source, etc.).
  - Example 5: Discuss bifurcation diagram for Logistic model with havesting (page 8)

$$x' = x(1-x) - h, \quad h > 0.$$

    * Equilibrium: $x(1-x) - h = 0$

$$x^2 - x + h = 0, \quad x = \frac{1 \pm \sqrt{1-4h}}{2}$$

    * So when $1 - 4h > 0$, or $h < 1/4$, there are two equilibrium solutions $0 < x_1 < 1/2 < x_2$
    * Bifurcation diagram: $x^2 - x + h = 0$



    * For each h, there are two solutions:

- $x_1(h)$ is the lower half of the parabola (source)
- This is because $f'(x) = 1 - 2x$. So $f'(x_1) = 1 - 2x_1 > 0$
- $x_2(h)$ is the upper half of the parabola (sink)
- $f'(x_2) = 1 - 2x_2 < 0$ (sink)

* when $h = 1/4$, there is only one equilibrium solution $x_0 = 1/2$.
* Moreover, $f'(x_0) = 0$, $f''(x_0) < 0$. So $f \leq 0$, it is a saddle
* when $1 - 4h < 0$, or $h > 1/4$, there is no equilibrium solution.
* Apparently, when across $h = 1/4$, the behavior of solutions change. It is a bifurcation.

- Poincaré Map
- Consider ODE with periodic structures

$$x' = f(t, x)$$

- where $f(t, x) = f(t + T, x)$ has period T.
- For instance, $x' = f(t, x) = ax(1 - x) - h(1 + \sin(2\pi t))$
  * $f(t, x)$ is periodic with $T = 1$ for variable t
- In this periodic case, if we can find solutions for t in one period $[0, T]$, then we can construct solutions for all times:
  * Let $x_1(t)$ be a solution for t in $[0, T]$. We then solve IVP

$$\begin{aligned} x' &= f(t, x) \\ x(0) &= x_1(T) \end{aligned}$$

  * Denote the solution as $x_2(t) : x_2(0) = x_1(T)$
  * Combine x_1 and x_2:

$$x(t) = \begin{cases} x_1(t) & \text{if } \ 0 \leq t \leq T \\ x_2(t - T) & \text{if } T \leq t \leq 2T \end{cases}$$

  * We can verify $x(t)$ also solves ODE in $[T,\ 2T]$:
    - Since $x(t) = x_2(t - T)$,
    - $x'(t) = x_2'(t - T) = f(t - T, x_2(t - T)) = f(t, x_2(t - T)) = f(t, x(t))$

- * Repeat this process: solve x_3 by

$$x' = f(t,x)$$
$$x(0) = x_2(T)$$

- * then solve x_{n+1} by

$$x' = f(t,x)$$
$$x(0) = x_n(T)$$

- * Construct $x(t)$ as

$$x(t) = \begin{cases} x_1(t) & \text{if } \quad 0 \le t \le T \\ x_{n+1}(t-T) & \text{if } nT \le t \le (n+1)T \end{cases}, \quad n = 0, 1, ...$$

- – So, solutions in $[0,T]$ play a significant role in studying ODEs

- For any $x_{0,}$ we solve IVP

$$x' = f(t,x)$$
$$x(0) = x_0$$

  - – and compute $x(T)$. This value depends on x_0. The map $x_0 \longmapsto x(T)$ is called the Poincaré map
  - – Poincaré map is denoted by $p(x_0) = x(T)$
  - – If $p(x_0) = x_0$, i.e., $x(1) = x_0 = x(0)$, the solution $x(t)$ is periodic with period T.
  - – So: a solution with initial value $x(0) = x_0$ iff $p(x_0) = x_0$ (or x_0 is a fixed point)

- Flows $\phi(t,x_0)$ of ODE
  - – In general, for any $x_{0,}$ we solve IVP

$$x' = f(t,x)$$
$$x(0) = x_0$$

  - – Solution is denoted as

$$x = \phi(t,x_0)$$

- this is a two variable function $(t, x_0) \longmapsto \phi(t, x_0)$
- We call it the flow associated with ODE.

- Example 6:
  - (a) for $x' = ax$, $\phi(t, x_0) = x_0 e^{at}$
  - (b) for $x' = ax(1-x)$, $\phi(t, x_0) = x_0 e^{at} / (1 - x_0 + x_0 e^{at})$

- When $f(t, x)$ is periodic in t, then

$$p(x_0) = \phi(T, x_0)$$

  is called the Poincaré map.

- Looking for periodic solutions = finding pixed points of the Poincaré map.

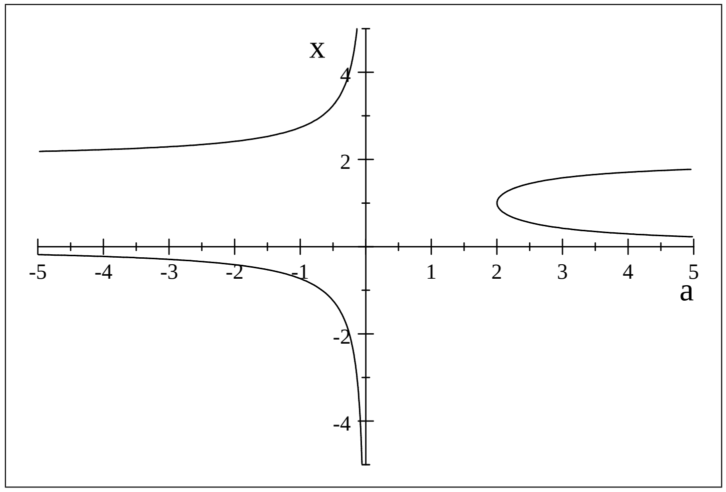
- Homework (due 9/16, before class starts)
  - Exercise#1 Find general solution for

$$x' = ax\left(1 - \frac{x}{N}\right) - h$$

    (You need to discuss different cases for various constants a, N, h. Solutions may be different depending on certain relationship between these constants.)
  - Exercise#2 For the above legistic model with harvesting, $N = 2, h = 1$, i.e.,

$$x' = ax\left(1 - \frac{x}{2}\right) - 1$$

    discuss its bifurcation and draw its bifurcation diagram. (hint: the graph of $ax\left(1 - \frac{x}{2}\right) - 1 = 0$ is shown below)



– From textbook: #2ace, 3ac, 4, 6, 10