Section 4.4 Non-homogeneous Heat Equation

- Homogenizing boundary conditions

Consider initial-Dirichlet boundary value problem of non-homogeneous heat equation

$$\begin{aligned} v_t - kv_{xx} &= F, \quad 0 < x < L, \; t > 0 \\ v(0,t) &= a(t) \\ v(L,t) &= b(t) \\ v(x,0) &= f(x). \end{aligned} \tag{1}$$

We first introduce new function

$$\begin{aligned} u(x,t) &= v(x,t) - \frac{a(t)(L-x) + b(t)x}{L} \\ &= v(x,t) + G(x,t), \end{aligned}$$

where

$$G(x,t) = -\frac{a(t)(L-x) + b(t)x}{L}$$

satisfies

$$\begin{aligned} G(0,t) &= -a(t) \\ G(L,t) &= -b(t) \\ G_{xx}(x,t) &= 0. \end{aligned}$$

So $u(x,t)$ satisfies the homogeneous boundary conditions

$$\begin{aligned} u(0,t) &= v(0,t) - G(0,t) = a(t) - a(t) = 0 \\ u(L,t) &= v(L,t) - G(L,t) = b(t) - b(t) = 0, \end{aligned}$$

and the heat equation

$$\begin{aligned} u_t - ku_{xx} &= v_t - kv_{xx} + (G_t - kG_{xx}) \\ &= F + G_t = H, \end{aligned}$$

where

$$\begin{aligned} H &= F + G_t \\ &= F - \frac{a'(t)(L-x) + b'(t)x}{L}. \end{aligned}$$

In other words, the heat equation (1) with non-homogeneous Dirichlet boundary conditions can be reduced to another heat equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} u_t - ku_{xx} &= H(x,t) \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= g(x) \end{aligned} \tag{2}$$

where

$$\begin{aligned} g(x) &= v(x,0) + G(x,0) \\ &= f(x) - \frac{a(0)(L-x) + b(0)x}{L} \\ &= f(x) - \frac{f(0)(L-x) + f(L)x}{L} \end{aligned}$$

and the compatibility conditions

$$a(0) = f(0), \;\; b(0) = f(L)$$

are applied here.

The same technique can be used to homogenize other types of boundary conditions (see homework).

- Homogenizing initial condition

We consider the heat equation with homogeneous Dirichlet boundary conditions (2)

$$\begin{aligned} u_t - ku_{xx} &= H(x,t) \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= g(x). \end{aligned}$$

By linearity, the solution is the sum of following two problems:

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= g(x), \end{aligned} \tag{3a}$$

and

$$\begin{aligned} u_t - ku_{xx} &= H \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= 0. \end{aligned} \tag{4}$$

The first problem (3a) can be solved by the method of separation of variables developed in section 4.1. So it remains to solve problem (4). If we can solve (4), then the original non-homogeneous heat equation (1) can be easily recovered.

- Solving non-homogeneous heat equation with homogeneous initial and boundary conditions.

We can now focus on (4)

$$\begin{aligned} u_t - ku_{xx} &= H \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= 0, \end{aligned}$$

and apply the idea of separable solutions.

Suppose $H(x,t)$ is piecewise smooth. It then has, for any fixed t, the Fourier series expansion

$$\begin{aligned} H(x,t) &= \sum_{n=1}^{\infty} \left(H_n(t) \sin \frac{n\pi x}{L} \right), \\ H_n(t) &= \frac{2}{L} \int_0^L H(x,t) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

We seek for a solution in the form

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{L}, \\ u_n(t) &= \frac{2}{L} \int_0^L u(x,t) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

This is actually the Fourier Sine series expansion for the solution with fixed t. By direct calculation,

$$\begin{aligned}
u_n'(t) &= \frac{2}{L}\int_0^L u_t(x,t)\sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L}\int_0^L (ku_{xx}(x,t) + H(x,t))\sin\frac{n\pi x}{L}dx \\
&= \frac{2k}{L}\int_0^L u_{xx}(x,t)\sin\frac{n\pi x}{L}dx + H_n(t).
\end{aligned}$$

Applying integration by parts twice, we have

$$\begin{aligned}
\int_0^L u_{xx}(x,t)\sin\frac{n\pi x}{L}dx &= \int_0^L \sin\frac{n\pi x}{L}du_x(x,t) \\
&= u_x(x,t)\sin\frac{n\pi x}{L}|_{x=0}^{x=L} - \int_0^L u_x(x,t)\,d\sin\frac{n\pi x}{L} \\
&= -\frac{n\pi}{L}\int_0^L u_x(x,t)\cos\frac{n\pi x}{L}dx \\
&= -\frac{n\pi}{L}\int_0^L \cos\frac{n\pi x}{L}du(x,t) \\
&= -\frac{n\pi}{L}\left[u(x,t)\cos\frac{n\pi x}{L}\right]_{x=0}^{x=L} + \frac{n\pi}{L}\int_0^L u(x,t)\,d\cos\frac{n\pi x}{L} \\
&= -\left(\frac{n\pi}{L}\right)^2\int_0^L u(x,t)\sin\frac{n\pi x}{L}dx \\
&= -\frac{(n\pi)^2}{2L}u_n(t).
\end{aligned}$$

Substituting this into the previous formula, we derive at

$$\begin{aligned}
u_n'(t) &= \frac{2k}{L}\int_0^L u_{xx}(x,t)\sin\frac{n\pi x}{L}dx + H_n(t) \\
&= \frac{2k}{L}\left(-\frac{(n\pi)^2}{2L}u_n(t)\right) + H_n(t) \\
&= -\frac{k(n\pi)^2}{L^2}u_n(t) + H_n(t).
\end{aligned}$$

The solution of this ODE with initial condition $u_n(0) = 0$ is

$$u_n(t) = \int_0^t e^{-\frac{k(n\pi)^2}{L^2}(t-\tau)} H_n(\tau)\, d\tau.$$

Substituting this into the Fourier series expansion for u, we conclude that the solution for the non-homogeneous heat equation with homogeneous initial and boundary conditions is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty} \int_0^t e^{-\frac{k(n\pi)^2}{L^2}(t-\tau)} H_n(\tau)\, d\tau \sin \frac{n\pi x}{L}. \end{aligned}$$

Homework:

1. Homogenize boundary conditions:

   (a) The Neumann boundary conditions

   $$\begin{aligned} u_t - k u_{xx} &= F(x,t) \\ u_x(0,t) &= a(t) \\ u_x(L,t) &= b(t) \\ u(x,0) &= f(x). \end{aligned}$$

   (b) Mixed boundary conditions

   $$\begin{aligned} u_t - k u_{xx} &= F(x,t) \\ u_x(0,t) &= a(t) \\ u(L,t) &= b(t) \\ u(x,0) &= f(x). \end{aligned}$$

2. Consider

   $$\begin{aligned} u_t - k u_{xx} &= H(x,t) \\ u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \\ u(x,0) &= 0, \end{aligned}$$

and suppose that

$$H(x,t) = \frac{H_0(t)}{2} + \sum_{n=1}^{\infty} \left(H_n(t) \cos \frac{n\pi x}{L} \right),$$

$$H_n(t) = \frac{2}{L} \int_0^L H(x,t) \cos \frac{n\pi x}{L} dx, \quad n \geq 0.$$

Find a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \cos \frac{n\pi x}{L},$$

$$u_n(t) = \frac{2}{L} \int_0^L u(x,t) \cos \frac{n\pi x}{L} dx.$$