Lecture 2. Solving Linear Systems

As we discussed before, we can solve any system of linear equations by the method of elimination, which is equivalent to applying a sequence of elementary row reductions over its augmented matrix. Some terminologies:

• Leading entry – the first nonzero entry of a row (or column) is called the leading entry of the row (or column).

• Echelon Form – A matrix is called in Echelon form (upper triangle form) if:

  (a) All non-zero rows are above any zero-row (row with all entries zero)

  (b) For any two rows, the column containing the leading entry of the upper row is on the left of the column containing the leading entry of the lower row.

In short, Echelon form displays upper-triangle pattern. In an Echelon form, leading entries of rows form a stair-shape. For instance (where * indicates a leading entry of a row),

Echelon form:

\[
\begin{bmatrix}
1^* & -2 & 1 & 3 & 0 \\
0 & 0 & -8^* & 1 & 8 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1^* & -2 & 1 & 3 & 0 \\
0 & 4^* & 1 & 1 & 8 \\
0 & 0 & 0 & 4^* & 0
\end{bmatrix}
\]

non-Echelon form:

\[
\begin{bmatrix}
0 & -2^* & 1 & 3 & 0 \\
2^* & 0 & -8 & 1 & 8 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

• Reduced Echelon Form – a matrix is called Reduced Echelon form If, in addition to (a) and (b) above,

  (c) all leading entries are 1 and it is the only non-zero entry in each pivot column.

For instance, the followings are Reduced Echelon forms:

\[
\begin{bmatrix}
1 & 0 & 0 & 3 & 0 \\
0 & 0 & 1 & 1 & 8 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 8 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

• Pivot – leading entry 1 in the reduced echelon form is called pivot position (or pivot entry.)

• Pivot Column – column containing a pivot position is called pivot column.
**Theorem 2.1.** Any matrix can be reduced by a sequence of elementary row operations to a unique reduced Echelon form.

**Theorem 2.2.** Elementary row operations do not affect the solution set of any linear system. Consequently, the solution set of a system is the same as that of the system whose augmented matrix is in the reduced Echelon form. The system can be solved from bottom up once it is reduced to an Echelon form.

*Gauss-Jordan Algorithm*

Gauss-Jordan Algorithm is a classic algorithm of finding reduced Echelon form. It consists of several steps described as follows.

Step 1. From the left, find the first non-zero column (it is a pivot column). By interchanging two rows if necessary, make sure that the leading entry of the column is non-zero. In other words, by performing row operations, we end up with the following matrix (* represents any number)

\[
\begin{bmatrix}
a & * & * & \ldots \\
b & * & * & \ldots \\
\vdots & \vdots & \vdots & \ldots \\
c & * & * & \ldots \\
\end{bmatrix}
\text{ or }
\begin{bmatrix}
0 & \ldots & 0 & a & * & \ldots \\
0 & \ldots & 0 & b & * & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & c & * & \ldots \\
\end{bmatrix}, \ a \neq 0.
\]

Step 2. Perform elementary row operation #3: \( R_1/a \to R_1 \), we arrive at

\[
\begin{bmatrix}
1 & \ast/a & \ast/a & \ldots \\
b & * & * & \ldots \\
\vdots & \vdots & \vdots & \ldots \\
c & * & * & \ldots \\
\end{bmatrix}.
\]

Then, perform elementary row operation #1 several times (i.e., first, \( R_2 - bR_1 \to R_2 \), ..., finally \( R_m - cR_1 \to R_m \)), we obtain a matrix

\[
\begin{bmatrix}
1 & \ast & \ast & \ldots \\
0 & A & B & \ldots \\
\vdots & \vdots & \vdots & \ldots \\
0 & C & * & \ldots \\
\end{bmatrix}.
\]

Step 3. Repeat the above two steps for the submatrix obtained from deleting the first row,

\[
\begin{bmatrix}
0 & A & B & \ldots \\
\vdots & \vdots & \vdots & \ldots \\
0 & C & * & \ldots \\
\end{bmatrix}.
\]

This sub-matrix is then reduced row-equivalently to

\[
\begin{bmatrix}
0 & 1 & E & * & \ldots \\
0 & 0 & * & \ldots \\
\vdots & \vdots & \vdots & \ldots \\
0 & 0 & G & * & \ldots \\
\end{bmatrix},
\]
and the original matrix is reduced to
\[
\begin{bmatrix}
1 & W & * & \\
0 & 1 & E & \\
\vdots & \vdots & \vdots & \\
0 & 0 & G & \\
\end{bmatrix}
\]

By perform row operation: \( R_1 \rightarrow W R_2 \rightarrow R_1 \), we have
\[
\begin{bmatrix}
1 & 0 & 0 & * \\
0 & 1 & * & * \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & * & * \\
\end{bmatrix}
\]

Step 4. Repeat till we are forced to stop – this final matrix is the reduced Echelon form.

**Example 2.1.** Find the reduced Echelon form for
\[
\begin{bmatrix}
1 & -2 & 1 & | & 0 \\
0 & 2 & -8 & | & 8 \\
-4 & 5 & 9 & | & -9 \\
\end{bmatrix}
\]

and then solve the corresponding linear system
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

Solution: We perform row operations on the augmented matrix as follows:

\[
\begin{align*}
\begin{bmatrix}
1 & -2 & 1 & | & 0 \\
0 & 2 & -8 & | & 8 \\
-4 & 5 & 9 & | & -9 \\
\end{bmatrix}
& \rightarrow
\begin{bmatrix}
1 & -2 & 1 & | & 0 \\
0 & 2 & -8 & | & 8 \\
0 & -3 & 13 & | & -9 \\
\end{bmatrix}
& \rightarrow
\begin{bmatrix}
1 & 0 & -7 & | & 8 \\
0 & 1 & -4 & | & 4 \\
0 & -3 & 13 & | & -9 \\
\end{bmatrix}
& \rightarrow
\begin{bmatrix}
1 & 0 & -7 & | & 8 \\
0 & 1 & -4 & | & 4 \\
0 & 0 & 1 & | & 3 \\
\end{bmatrix}
& \rightarrow
\begin{bmatrix}
1 & 0 & 0 & | & 29 \\
0 & 1 & 0 & | & 16 \\
0 & 0 & 1 & | & 3 \\
\end{bmatrix}
\end{align*}
\]

We obtain the reduced Echelon form
\[
\begin{bmatrix}
1 & 0 & 0 & | & 29 \\
0 & 1 & 0 & | & 16 \\
0 & 0 & 1 & | & 3 \\
\end{bmatrix}
\]
The linear system with the above matrix as its augmented matrix is

\[
\begin{align*}
\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{array} & \Rightarrow & \begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{array} \\
R_3 + 4R_1 & \Rightarrow & \begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{array} & \Rightarrow & \begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & 0 & 1 & 3
\end{array}.
\end{align*}
\]

Next, we write down the corresponding system

\[
\begin{align*}
\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{array} & \Rightarrow & \begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & 0 & 1 & 3
\end{array} \\
R_3 - 3R_1 & \Rightarrow & \begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{array} & \Rightarrow & \begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & 0 & 1 & 3
\end{array}.
\end{align*}
\]

Finally, we solve the system bottom up: From the last equation, we see

\[x_3 = 3.\]

Next, we substitute this into the second equation to get

\[x_2 - 4 \cdot 3 = 4 \implies x_2 = 16.\]

We then substitute \(x_3 = 3\) and \(x_2 = 16\) into the first equation, we obtain

\[x_1 - 2 \cdot 16 + 3 = 0 \implies x_1 = 29.\]

In summary, to solve a system of linear equations whose augmented matrix is \(M\). We follow the following steps.

**Step 1:** Perform elementary row operations to reduced \(M\) to its reduced Echelon form \(E\).

**Step 2:** In the matrix \(E\), locate pivot positions and pivot columns. The variable associated with pivot columns are called **basic variables**. For instance, for

\[
E = \begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 5 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\(x_1\) and \(x_2\) are basic variables.
Step 3: Write down the system of linear equations whose augmented matrix is $E$. Solve all pivot variables in terms of non-pivot variables. Those non-basic variables are referred as to free variables or parameters since they can be chosen freely to generate solutions.

It is possible to solve a linear system using any Echelon form instead of its reduced Echelon form. To do so, the very last step needs to be modified to

Step 3’: Write down the system of linear equations whose augmented matrix is $E$. Solve all pivot variables in terms of non-pivot variables by backward substitutions.

Example 2.2. Consider a linear system whose augmented matrix is reduced to an Echelon form

$$
\begin{bmatrix}
1 & 3 & -1 & | & 1 \\
0 & 1 & 2 & | & 2 \\
0 & 0 & 0 & | & 0
\end{bmatrix}.
$$

Find the describe the solution set.

Solution: Notice that the matrix has two pivot columns that are associated with variables $x_1$ and $x_2$. These two variable are basic variables. $x_3$ is the free variable. We write down the corresponding system

$$
x_1 + 3x_2 - x_3 = 1 \\
x_2 + 2x_3 = 2,
$$

and solve, from bottom up, basic variables in terms of the free variable. In this case, from the second equation, we solve $x_2$ in terms of $x_3$:

$$
x_2 = 2 - 2x_3.
$$

Next, we substitute it into the first equation

$$
x_1 + 3(2 - 2x_3) - x_3 = 1
$$

and solve for $x_1$ as

$$
x_1 = 13 - (2 - 2x_3) + x_3 = 11 + 3x_3.
$$

Here, $x_3$ can be chosen arbitrarily: for any $x_3 = t$, we have a solution

$$x_1 = 11 + 3t \\
x_2 = 2 - 2t \\
x_3 = t.
$$

(1)

Therefore, there are infinite many solitons. For instance, take $t = 0$,

$$x_1 = 11, \; x_2 = 2, \; x_3 = 0;$$

take $t = 1$

$$x_1 = 14, \; x_2 = 0, \; x_3 = 1.$$
We call (1) a parametric representation of the solution set, or general solution. In general, we choose non-pivot variables as parameters and solve pivot variables in terms of non-pivot variables.

Example 2.3 Solve

\[
\begin{align*}
    x_1 - 2x_2 + x_3 + x_4 &= 0 \\
    2x_2 + 6x_4 &= 4
\end{align*}
\]

and describe general solution by a parametric representation.

Solution: Its augmented matrix

\[
\begin{bmatrix}
    1 & -2 & 1 & 1 & | & 0 \\
    0 & 2 & 0 & 6 & | & 4
\end{bmatrix}
\]

is already in Echelon form, and \( x_1 \) and \( x_2 \) are pivot variables. So we first solve \( x_2 \) from the second equation as

\[
x_2 = 2 - 3x_4.
\]

Then substitute it into the first equation and solve for \( x_1 \) as

\[
x_1 = 2 (2 - 3x_4) + x_3 + x_4 = 0
\]

\[
x_1 = 2 (2 - 3x_4) - x_3 - x_4 = 4 - x_3 - 7x_4.
\]

Set \( x_3 = s, x_4 = t \), we find a parametric representation

\[
\begin{align*}
    x_1 &= 4 - s - 7t \\
    x_2 &= 2 - 3t \\
    x_3 &= s \\
    x_4 &= t
\end{align*}
\]

- Consistency

Not every system has a solution. For instance, consider a system of three equation with three unknowns. In 3D space, each linear equation represents a plane. So the system represents the intersection of three planes, which could be a single point, or a straight line, or a plane (all three planes are identical), or empty set (no solution).

Theorem 2.3 (Consistency) A system is called consistent if it admits at least one solution. A system is consistent if and only if the rightmost column contains NO pivot.

Proof: If the rightmost column is a pivot column, then in its reduced Echelon form, there is a pivot in the rightmost column:

\[
\begin{bmatrix}
    1 & * & \ldots & * & | & * \\
    0 & * & \ldots & * & | & * \\
    0 & 0 & \ldots & A & | & b \\
    0 & 0 & 0 & 0 & | & 0
\end{bmatrix}, \quad A = 0.
\]
In other words, there is a row (the last not-all-zero row) having the form 
\[0, 0, \ldots, 0, b\], where \(b \neq 0\). Accordingly, the very last non-trivial equation is

\[0 = b, \quad \text{but} \quad b \neq 0,\]

A contradiction.

**Example 2.4.** In the second example of the previous section, we have

\[
\begin{bmatrix}
0 & 1 & -4 & | & 8 \\
2 & -3 & 2 & | & 1 \\
5 & -8 & 7 & | & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3/2 & 1 & | & 1/2 \\
0 & 1 & -4 & | & 8 \\
0 & 0 & 0 & | & 5/2
\end{bmatrix}.
\]

The last column is a pivot column. Thus, it is inconsistent.

**Example 2.5.** Determine if the system

\[
\begin{align*}
3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15
\end{align*}
\]

Solution. The augmented matrix is

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & | & -5 \\
3 & -7 & 8 & -5 & 8 & | & 9 \\
3 & -9 & 12 & -9 & 6 & | & 15
\end{bmatrix}
\]

We perform following row operations:

\[
R_2 \rightarrow R_1
\]

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & | & -5 \\
3 & -7 & 8 & -5 & 8 & | & 9 \\
3 & -9 & 12 & -9 & 6 & | & 15
\end{bmatrix}
\]

\[
R_3 - R_1 \rightarrow R_3
\]

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & | & -5 \\
3 & -7 & 8 & -5 & 8 & | & 9 \\
0 & -2 & 4 & -4 & -2 & | & 6
\end{bmatrix}
\]

\[
R_3 + \frac{2}{3} R_2 \rightarrow R_3
\]

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & | & -5 \\
3 & -7 & 8 & -5 & 8 & | & 9 \\
0 & 0 & 0 & 2/3 & * & |
\end{bmatrix}
\]

We conclude that the linear system is consistent.

When a linear system is consistence, from the above examples, we see that
the system admits a unique solution if and only if there is no free variable.

**Theorem 2.4.** Consider a linear system whose augmented matrix is \(M\).
There are three possibilities. (1) If the last column of \(M\) is a pivot column, then
the system is inconsistent. (2) If the last column of \(M\) is a not pivot column
but the rest columns are all pivot, then the system admits a unique solution.
(3) If, additional to the last column that is not pivot, there is at least one more non-pivot column, then the system has infinite many solutions.

Note that Gauss-Jordan Algorithm is the most basic algorithm. In Matlab or Mathematica, there are several build-in functions for solving linear systems. There are many more efficient algorithm for solving linear systems. Here we introduce another popular algorithm.

**LU Decomposition**

Another classical method to solve linear system $A\vec{x} = \vec{b}$ is so-called LU decomposition: Find a lower triangle matrix $L$ and a upper triangle matrix $U$ such that

$$A_{m \times n} = L_{m \times n} U_{m \times n} = \begin{bmatrix}1 & 0 & \cdots & 0 \\ \ast & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \cdots & 1 \end{bmatrix} \begin{bmatrix}a_{11} & \ast & \cdots & \ast \\ 0 & a_{22} & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \ast \end{bmatrix}$$

Now the linear system $A\vec{x} = \vec{b}$ can be solved by solving two easier systems

$$L\vec{y} = \vec{b}$$
$$U\vec{x} = \vec{y}.$$ 

The first system $L\vec{y} = \vec{b}$ can be solved from top to bottom, and the second system can be solved from bottom up.

**LU Decomposition Algorithm**

1. Reduce $A$ to an echelon form from $U$ by a sequence of type one row operations (row replacement row operation)

2. Place entries in $L$ such that the same sequence of row operations reduces $L$ to the identity matrix.

The rigorous justification of this algorithm will be presented later in Lecture 7.

Example 2. Find LU decomposition and then solve for $A\vec{x} = \vec{b}$:

$$A = \begin{bmatrix}2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix}1 \\ 0 \\ 2 \\ 14 \end{bmatrix}.$$ 

Solution: Obviously, since $A$ is $4 \times 5$, $L$ should be $4 \times 4$. The first step to reduce $A$ to an echelon form is to replace the first column as follows

$$A = \begin{bmatrix}2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & 3 & 1 \end{bmatrix}$$

$$\begin{array}{lr}2R_1 + R_2 \rightarrow R_2 \\ -R_1 + R_3 \rightarrow R_3 \\ 3R_1 + R_4 \rightarrow R_4 \end{array}$$

$$\begin{bmatrix}2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1.$$
Therefore,

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
1 & a & 1 & 0 \\
-3 & b & c & 1
\end{bmatrix}
\overset{2R_1 + R_2 \rightarrow R_2}{\longrightarrow}
\overset{-R_1 + R_3 \rightarrow R_3}{\longrightarrow}
\overset{3R_1 + R_4 \rightarrow R_4}{\longrightarrow}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & a & 1 & 0 \\
0 & b & c & 1
\end{bmatrix} = L_1.
\]

We repeat the process to find \(a, b\):

\[
A_1 = \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & -9 & -3 & -4 & 10 \\
0 & 12 & 4 & 12 & -5
\end{bmatrix}
\overset{3R_2 + R_3 \rightarrow R_3}{\longrightarrow}
\overset{-4R_2 + R_4 \rightarrow R_4}{\longrightarrow}
\begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 4 & 7
\end{bmatrix} = A_2
\]

and accordingly \(a = -3, b = 4\) because

\[
L_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -3 & 1 & 0 \\
0 & 4 & c & 1
\end{bmatrix}
\overset{3R_2 + R_3 \rightarrow R_3}{\longrightarrow}
\overset{-4R_2 + R_4 \rightarrow R_4}{\longrightarrow}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & c & 1
\end{bmatrix} = L_2.
\]

Finally, we find \(c\):

\[
A_2 = \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 4 & 7
\end{bmatrix}
\overset{-2R_3 + R_4 \rightarrow R_4}{\longrightarrow}
\begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix} = A_2
\]

so \(c = 2\) since

\[
L_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 2 & 1
\end{bmatrix}
\overset{-2R_3 + R_4 \rightarrow R_4}{\longrightarrow}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

We conclude

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
1 & -3 & 1 & 0 \\
-3 & 4 & 2 & 1
\end{bmatrix},
U = \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}.
\]

Now we solve

\[
L\tilde{y} = \begin{bmatrix}
1 & 0 & 0 & 0 & y_1 \\
-2 & 1 & 0 & 0 & y_2 \\
1 & -3 & 1 & 0 & y_3 \\
-3 & 4 & 2 & 1 & y_4
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
2 \\
14
\end{bmatrix}.
\]
\[ \begin{align*}
y_1 &= 1 \\
-2y_1 + y_2 &= 0 \implies y_2 = 2y_1 = 2 \\
y_1 - 3y_2 + y_3 &= 2 \implies y_3 = 2 - y_1 + 3y_2 = 2 - 1 + 6 = 7 \\
-3y_1 + 4y_2 + 2y_3 + y_4 &= 14 \implies y_4 = 14 + 3y_1 - 4y_2 - 2y_3 = 14 + 3 - 8 - 14 = -5.
\end{align*} \]

Finally, we solve
\[
\begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
2 \\
7 \\
-5 \\
\end{bmatrix}
\]
from bottom up (\(x_3\) is free variable):
\[
x_5 = -1
\]
\[
2x_4 + x_5 = 7 \implies x_4 = \frac{7 - x_5}{2} = 4
\]
\[
3x_2 + x_3 + 2x_4 - 3x_5 = 2 \implies x_2 = \frac{2 - x_3 - 2x_4 + 3x_5}{3}
\]
\[
= \frac{2 - x_3 - 8 - 3}{3} = -3 - \frac{x_3}{3}
\]
\[
2x_1 + 4x_2 - x_3 + 5x_4 - 2x_5 = 1 \implies x_1 = \frac{1 - 4x_2 + x_3 - 5x_4 + 2x_5}{2}
\]
\[
= \frac{1 - 4\left(-3 - \frac{x_3}{3}\right) + x_3 - 20 - 2}{2} = \frac{7}{6}x^3 - \frac{9}{2}.
\]

**Remark:** When there is a need to exchange two rows, the LU algorithm need to modified accordingly.

- **Homework #2**

1. Determine whether each of the following matrices is in reduced Echelon form or only in Echelon form, or neither. If in Echelon form, indicate all pivot positions and pivot columns.

\[
\begin{align*}
(a) & \begin{bmatrix}
0 & 1 & 0 & 6 & 0 & -5 \\
0 & 0 & 1 & -5 & 0 & 9 \\
0 & 0 & 0 & 0 & 1 & 15 \\
\end{bmatrix} \\
(b) & \begin{bmatrix}
1 & 1 & 2 & 6 & 0 & -5 \\
0 & 0 & 1 & -5 & 1 & 9 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
(c) & \begin{bmatrix}
1 & 1 & 2 & 6 & 0 \\
0 & 0 & 1 & -5 & 1 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]
2. Determine the value(s) of $h$ such that the matrix is the augmented matrix of a consistent linear system. Then find the general solution.

$$\begin{bmatrix} 1 & 1 & 2 & 6 & 0 \\ -1 & 0 & 1 & -5 & h \\ 3 & 1 & 0 & 16 & 2 \end{bmatrix}.$$

3. Find general solutions for linear systems whose augmented matrices are given:

(a) $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}.$$

(b) $$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

4. Find LU Decomposition for

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -6 & 6 \end{bmatrix}$$

make sure verify your answer by direct multiplication of $LU$.

5. For each of the following statements, determine whether it is true or false. If your answer is true, state your rationale. If false, provide an counterexample (the example contradicting the statement).

(a) A matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

(b) A basic variable is a variable corresponding to a pivot columns

(c) If one row in an echelon form of an augmented matrix is $[0 \ 0 \ 0 \ 0 \ 0 \ 3 \ 0]$, then the associated system is inconsistent.

(d) The pivot positions depend on how you do row reduction.

(e) Whenever a system have more than one solution, it must has a free variable.