1 Section 6.1: Inner Product, Length and Orthogonality

Definition. Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

be two vectors in R^n. The inner product, denoted by $\vec{u} \cdot \vec{v}$, is defined as

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = [u_1, u_2, ..., u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + ... + u_n v_n.$$

Example 6.1.1. Let

$$\vec{u} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Then

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 3 & -5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3 \cdot 2 - 5 \cdot 2 + 1 \cdot 0 + 2 \cdot (-1) = -1.$$

Properties of Inner Product:

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, (b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$, (c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$, (d) $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = 0$.

Definition. Length (or magnitude) of a vector $\vec{u}$, denoted by $\|\vec{u}\|$, is defined as

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}.$$

Example 6.1.2. For the same $\vec{u}$ as in the previous example,

$$\|\vec{u}\| = \sqrt{3^2 + (-5)^2 + 1^2 + 2^2} = \sqrt{39}.$$

Properties (continues): (e) $\|c\vec{u}\| = |c| \|\vec{u}\|$.

Example 6.1.3. For the same $\vec{u}$ as in the previous example,

$$\|-2\vec{u}\| = \sqrt{(-6)^2 + (10)^2 + (-2)^2 + (-4)^2} = \sqrt{156} = 2\sqrt{39} = 2\|\vec{u}\|.$$

Definition. Distance between two vectors $\vec{u}$ and $\vec{v}$, denoted by $dist\,(\vec{u},\vec{v})$, is defined as

$$dist\,(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+...+(u_n-v_n)^2}.$$

Example 6.1.4. Let

$$\vec{u} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Then,

$$dist\,(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \left\| \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ -6 \\ 1 \\ 3 \end{bmatrix} \right\| = \sqrt{47}.$$

Definition. The angle between two vectors $\vec{u}$ and $\vec{v}$, denoted by $\langle\vec{u},\vec{v}\rangle$, is defined by

$$\cos\langle\vec{u},\vec{v}\rangle = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad 0 \le \langle\vec{u},\vec{v}\rangle \le \pi.$$

Two vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal to each other if $\langle\vec{u},\vec{v}\rangle = \pi/2$, or $\vec{u}\cdot\vec{v} = 0$. We use the notation $\vec{u}\perp\vec{v}$ when $\vec{u}$ and $\vec{v}$ are orthogonal.

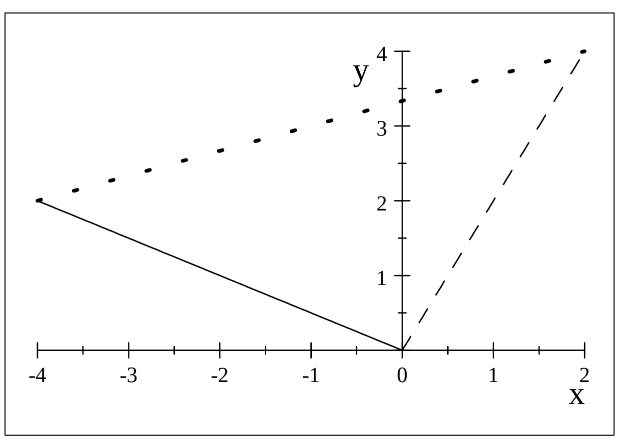
Example 6.1.5. For the same $\vec{u}$ and $\vec{v}$ as in the previous example, find the angle $\langle\vec{u},\vec{v}\rangle$.

Solution:

$$\cos\langle\vec{u},\vec{v}\rangle = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{-1}{\sqrt{39}\sqrt{4+1+1}} = -0.06\,5,$$
$$\langle\vec{u},\vec{v}\rangle = \arccos(-0.06\,5) = 1.635\,8(rad) = 93.724^o$$

Theorem (Pythagorean) In R^n, $\vec{u}\perp\vec{v}$ iff

$$\|\vec{u}\ -\ \vec{v}\|^2 = \|\vec{u}\ \|^2 + \|\ \vec{v}\|^2 \quad \text{or} \quad \|\vec{u}\ +\ \vec{v}\|^2 = \|\vec{u}\ \|^2 + \|\ \vec{v}\|^2.$$



solid line = $\vec{u}$, dash line $\vec{v}$, dot line = $\vec{u}+\vec{v}$

Proof. We verify using direct computation and the fact $\vec{u} \cdot \vec{v} = 0$:

$$\begin{aligned} LHS &= \|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 = RHS. \end{aligned}$$

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Definition. Let W be a subspace of R^n. A vector $\vec{u}$ is said to be orthogonal to W, denoted by $\vec{u} \perp W$, if $\vec{u}$ is orthogonal to every vector in W, i.e.,

$$\vec{u} \cdot \vec{w} = 0 \text{ for any } \vec{w} \in W \quad (\vec{w} \in W \text{ means } \vec{w} \text{ belongs to } W).$$

We call the subspace

$$W^{\perp} = \{\vec{v} \mid \vec{v} \perp W\}$$

the orthogonal complement space of W.

Example 6.1.6. In R^2, let W be a line passing through the origin. Then, its orthogonal complement, $W^{\perp}$ is the line passing through the origin and perpendicular to W. In R^3, (a) let let W be a line passing through the origin, then $W^{\perp}$ is the plane passing through the origin and perpendicular to W; (b) let W be a plane passing through the origin, then $W^{\perp}$ is the line passing through the origin and perpendicular to W.

Example 6.1.7. Let

$$W = Span \left\{ \vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -8 \\ -1 \end{bmatrix}. \right\}$$

Find and describe $W^{\perp}$.

Solution. We are looking for all $\vec{x}$ in R^4 such that

$$\begin{aligned} \vec{x} \cdot \vec{u} = 0 \\ \vec{x} \cdot \vec{v} = 0 \end{aligned}$$

or

$$\begin{bmatrix} \vec{x} \cdot \vec{u} \\ \vec{x} \cdot \vec{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let A be the matrix of rows $\vec{u}$ and $\vec{v}$, i.e.,

$$A = \begin{bmatrix} \vec{u}^T \\ \vec{v}^T \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & -8 & -1 \end{bmatrix}.$$

Then, $W^{\perp}$ consists of all vectors $\vec{x}$ such that

$$A\vec{x} = \begin{bmatrix} \vec{u}^T \\ \vec{v}^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{u} \cdot \vec{x} \\ \vec{v} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words,

$$W^{\perp} = Null\,(A)\,.$$

In general, let

$$W = Span\,\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_p\} \text{ be a subspace in } R^n.$$

Then

$$W^{\perp} = Null\,(A)\,, \quad A = \left([\vec{u}_1\ \vec{u}_2\ ...\ \vec{u}_p]_{n\times p}\right)^T = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_p^T \end{bmatrix}_{p\times n}.$$

We now proceed to describe $W^{\perp} = Null\,(A)$ using row operations.

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & -8 & -1 \end{bmatrix} \overset{R_2-2R_1\to R_2}{\longrightarrow} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 5 & -10 & -5 \end{bmatrix}$$

$$\overset{\substack{R_2/5 \to R_2 \\ R_1+2R_2\to R_1}}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & -1 \end{bmatrix}.$$

So, $x_1 = x_3 - x_4,\ x_2 = 2x_3 + x_4$. The parametric form is (with $x_3 = s,\ x_4 = t$)

$$\vec{x} = \begin{bmatrix} x_3 - x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

and

$$W^{\perp} = Null\,(A) = Span \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now let A be a $m \times n$ matrix that has m row vectors,

$$A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{bmatrix}, \quad \vec{v}_i \text{ be a row vector, i.e., } (\vec{v}_i)^T = \vec{u}_i \text{ is a column vector.}$$

Then, the "row space" of A is defined by

$$Row\,(A) = Span\,\{\text{rows of } A\} = Span\,\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_m\}\,,$$

where a vector is understood as a row, i.e., $1 \times n$ matrix. If we would like to interpret row vectors as column vectors, then

$$\begin{aligned} Row\,(A) &= (Span\,\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_m\})^T = Span\left\{(\vec{v}_1)^T, (\vec{v}_2)^T, ..., (\vec{v}_m)^T\right\} \\ &= Span\,\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_m\} \end{aligned}$$

From the above example, we can easily see that

$$Row\,(A)^{\perp} = Nul\left(\begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_m^T \end{bmatrix}_{m\times n}\right) = Nul\left(\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{bmatrix}_{m\times n}\right) = Null\,(A)\,.$$

Analogously, since columns of A = rows of A^T, or

$$Col\,(A) = Row\left(A^T\right),$$

we have

$$Col\,(A)^{\perp} = Row\left(A^T\right)^{\perp} = Null\left(A^T\right).$$