Section 4.5 & 4.6: Dimension & Rank

Consider subspaces of the form $H = Span\{\vec{v}_1, \vec{v}_{2,...}, \vec{v}_p\}$. When the generators $\vec{v}_1, \vec{v}_{2,...}, \vec{v}_p$ are linearly independent, they form a basis for H.

Theorem Let $B = \{\vec{v}_1, \vec{v}_{2,...}, \vec{v}_p\}$ be a basis of H. Then any set of more than p vectors in H is linearly dependent.

Proof: Let T be the coordinate mapping associated with this basis B, i.e., for any vector $\vec{x}$ in H, there is a unique expression

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_p\vec{v}_p.$$

Recall that this unique column vector of the linear relation is called the coordinate of $\vec{x}$ relative to the basis $\mathcal{B}$ (or $\mathcal{B}-$coordinate, in short), and is denoted by

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \quad \mathcal{B}-\text{coordinate of } \vec{x},$$

The coordinate mapping associated with this basis is defined as

$$T(\vec{x}) = [\vec{x}]_{\mathcal{B}} : \ H \to R^p.$$

Now, suppose $C = \{\vec{u}_1, \vec{u}_2, ..., \vec{u}_q\}$ be a subset of H with $q > p$. Then q vectors in R^p

$$T(\vec{u}_1), T(\vec{u}_2), ..., T(\vec{u}_q)$$

are linearly dependent. In other words, there is a nontrivial solution for

$$x_1T(\vec{u}_1) + x_2T(\vec{u}_2) + ... + x_qT(\vec{u}_q) = 0$$

or

$$T(x_1\vec{u}_1 + x_2\vec{u}_2 + ... + x_q\vec{u}_q) = 0.$$

Since T is one-to-one, this leads to

$$x_1\vec{u}_1 + x_2\vec{u}_2 + ... + x_q\vec{u}_q = 0.$$

Conclusion: From this Theorem, if H has two bases

$$B = \{\vec{v}_1, \vec{v}_{2,...}, \vec{v}_p\}, \quad C = \{\vec{u}_1, \vec{u}_2, ..., \vec{u}_q\}$$

then $p = q$.

However, a subspace may have infinite many sets of basis, for instance, all of the following sets are bases:

$$2\vec{v}_1, \vec{v}_{2,...}, \vec{v}_p$$
$$(\vec{v}_1 + 2\vec{v}_2 + 5\vec{v}_3), \vec{v}_{2,...}, \vec{v}_p.$$

Definition. Let H be a subspace of V with a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ of p linearly independent vectors. We call p, the number of vectors in a basis, the dimension of H, and denote it by

$$p = \dim(H).$$

Example. In R^n, the standard basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ consists of n vectors. So, $\dim(R^n) = n$. For any vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n, \ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Example. Consider R^2. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find (a) $[\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}}$ and $[\ \vec{e}_2]_{\{\vec{v}_1,\vec{v}_2\}}$, (b) $[\ \vec{x}]_{\{\vec{v}_1,\vec{v}_2\}}$ for any $\vec{x}$.

Solution: To find $[\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}}$, we need to solve $\vec{e}_1 = x\vec{v}_1 + y\ \vec{v}_2$, or

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We proceed by reducing the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \end{bmatrix}.$$

Therefore,

$$\vec{e}_1 = \frac{1}{3}\vec{v}_1 + \left(-\frac{2}{3}\right)\ \vec{v}_2, \quad [\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}.$$

This relation can be written as

$$\vec{e}_1 = \frac{1}{3}\vec{v}_1 + \left(-\frac{2}{3}\right)\ \vec{v}_2 = [\vec{v}_1,\ \vec{v}_2] \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix} = [\vec{v}_1,\ \vec{v}_2]\,[\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}}.$$

Similarly, by solving $\vec{e}_2 = x\vec{v}_1 + y\ \vec{v}_2$, or by row operations

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{bmatrix},$$

we found

$$[\ \vec{e}_2]_{\{\vec{v}_1,\vec{v}_2\}} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

We may also write the above relations as

$$\vec{e}_2 = x\vec{v}_1 + y\ \vec{v}_2 = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\ \vec{v}_2 = [\vec{v}_1,\ \vec{v}_2]\begin{bmatrix}1/3\\1/3\end{bmatrix} = [\vec{v}_1,\ \vec{v}_2]\,[\ \vec{e}_2]_{\{\vec{v}_1,\vec{v}_2\}}\,.$$

From this, we see that the 2×2 matrix with columns $\vec{e}_1, \vec{e}_2$,

$$[\vec{e}_1, \vec{e}_2] = \Big[[\vec{v}_1,\ \vec{v}_2]\,[\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}}\,, [\vec{v}_1,\ \vec{v}_2]\,[\ \vec{e}_2]_{\{\vec{v}_1,\vec{v}_2\}}\Big] = [\vec{v}_1,\ \vec{v}_2]\Big[[\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}}\,, [\ \vec{e}_2]_{\{\vec{v}_1,\vec{v}_2\}}\Big] = [\vec{v}_1,\ \vec{v}_2]\,V,$$

where the matrix

$$V = \Big[[\ \vec{e}_1]_{\{\vec{v}_1,\vec{v}_2\}}\,, [\ \vec{e}_2]_{\{\vec{v}_1,\vec{v}_2\}}\Big] = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

represents changes of coordinates from the standard basis to basis $\{\vec{v}_1, \vec{v}_2\}$. In fact, for any $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, we have

$$\begin{aligned}\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 &= x_1\left(\frac{1}{3}\vec{v}_1 + \left(-\frac{2}{3}\right)\ \vec{v}_2\right) + x_2\left(\frac{1}{3}\vec{v}_1 + \frac{1}{3}\ \vec{v}_2\right)\\ &= \left(\frac{1}{3}x_1 + \frac{1}{3}x_2\right)\vec{v}_1 + \left(\left(-\frac{2}{3}\right)x_1 + \frac{1}{3}x_2\right)\ \vec{v}_2.\end{aligned}$$

Using matrix notation,

$$\vec{x} = [\vec{e}_1, \vec{e}_2]\,\vec{x} = [\vec{v}_1,\ \vec{v}_2]\,V\vec{x}.$$

Hence,

$$[\vec{x}]_{\{\vec{v}_1,\vec{v}_2\}} = V\vec{x} = \begin{bmatrix}1/3 & 1/3\\ -2/3 & 1/3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{1}{3}x_1 + \frac{1}{3}x_2\\ \left(-\frac{2}{3}\right)x_1 + \frac{1}{3}x_2\end{bmatrix}.$$

Theorem: Let H be a subspace of finite dimension. Then any set of linearly independent vectors can be expanded into a basis.

Theorem (The Basis Theorem). Let $\dim V = p$. Then any set of p vectors that spans V automatically forms a basis.

Definition. The dimension of the column space of a matrix A is the same as RANK of the matrix A, i.e.,

$$Rank\,(A) = \dim\,(Col\,(A)) \quad (\text{ or simply } R\,(A)\quad).$$

As we explained in the previous lecture,

$$Col\,(A) = Span\,\{\text{pivot columns of } A\}\,.$$

Recall that $Rank\,(A) =$ number of pivots. So

$$\dim\,(Col\,(A)) = \text{number of pivots}$$

On the other hand, $Null\,(A)$ can be expressed using parametric vector forms, in which the number of free variables is equal to the number of independent generators. So

$$\dim\left(Null\,(A)\right) = \text{number of free variables} = \text{number of non-pivot columns}$$

Example. Find $Rank\,(A)$ and $\dim\left(Null\,(A)\right)$ if

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: We see that column#1, #2, #4 are pivot. Therefore, these three columns form a basis, and $Rank\,(A) = 3$. Since non-pivot columns correspond to free variables, in this case, there are two free variables in the solution set for $A\vec{x} = \vec{0}$, i.e., $\dim\left(Null\,(A)\right) = 2$.

The above example demonstrates the following **Dimension Theorem for** $A_{m\times n}$

$$Rank\,(A) + \dim\left(Null\,(A)\right) = n$$
$$(\text{number of pivots}) + (\text{number of non-pivot columns}) = \text{number of columns}.$$

Inverse Matrix Theorem (part 2) Let A be an $n \times n$ matrix. each of the following statements is equivalent to A is invertible:

1. The columns of A form a *basis* of R^n.
2. $Col\,(A) = R^n$.
3. $\dim(Col\,(A)) = n$
4. $Rank\,(A) = n$
5. $Null\,(A) = \{0\}$
6. $\dim Null\,(A) = 0$