# Section 4.1– 4.4: Vector Spaces and Subspaces

**Definition 1** *A vector space is a nonempty set $V$ of objects, called vectors, on which are defined two operations, called addition "+" and scalar multiplication, satisfying the following properties:*

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
3. There is a $\vec{0}$ in $V$ such that $\vec{u}+\vec{0}=\vec{0}+\vec{u}$
4. For any vector $\vec{u}$ in $V$, there is a vector $-\vec{u}$ such that $\vec{u}+(-\vec{u})=\vec{0}$
5. For any real number $c$, $c(\vec{u}+\vec{v})= c\vec{u}+c\vec{v}$
6. $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
7. $c(d\vec{u})=(cd)\vec{u}$
8. $1\vec{u}=\vec{u}$

Examples of vectors spaces

1. $R^n$ with usual addition and scalar multiplication
2. $M_{mn}=$ set of all $m\times n$ matrices
3. $P=$ set of all polynomials
4. $P_n=$ set of all polynomials with degree less than or equal to $n$
5. $C(a,b)=$ set of all continuous function
6. $C^n(a,b)=$ set of all continuous function with up to $n$th order continuous derivatives
7. All solutions of ODE $y'+p(t)y=0$
8. All solutions of ODE $y''+p(t)y'+q(t)y=0$

**Definition 2** *A subspace $W\subset V$ of a vector space $V$ is a subset that is closed under operations, i.e.,*
*(1)* $\vec{x},\vec{y}\in W \implies \vec{x}+\vec{y}\in W$, (2) $\vec{x}\in W$, $k$ *constant* $\implies k\vec{x}\in W$

**Example 3** *(1)* $\{0\}$ *and* $V$ *are always subspaces, called trivial subspaces*

*(2) The only other trivial subspace in* $R^n$. *In* $R^2$, *the only non-trivial subspaces are lines passing through the origin.*

*(3) In* $R^3$, *either a line passing through the origin or a plane passing through the origin is a subspace.*

*(4)Let* $A$ *be a matrix. The solution set of* $A\vec{x} = \vec{0}$ *is a subspace.*

*(5)* $Span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ *is a subspace.*

*(6) Any subspace in* $R^n$ *is a subspace spanned by no more than* $n$ *vectors.*

*(7)* $P_n \subset P \subset C^n(a,b) \subset C(a,b)$

**Definition 4** *A set of vectors* $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ *in* $V$ *is called linearly independent if the only solution of*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + ... + x_p\vec{v}_p = 0$$

*is* $x_1 = x_2 = ... = x_p = 0$.

***Theorem 5*** $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ *is linearly independent iff at least one is a linear combination of the rest.*

***Definition 6*** *Let*

$$H = Span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$$

*H be a subspace of V. We call* $B = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ *a basis for H if* $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ *is linearly independent.*

**Example 7** *In* $R^n$, *the columns of* $I_n$ *form the standard basis:*

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, ..., \ \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

*If* $\vec{x} = [x_1, x_2, ..., x_n]^T$, *then* $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 +, ..., x_n\vec{e}_n$.

**Theorem 8** *Suppose* $H = Span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$. *Then a subset of* $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ *form a basis.*

**Definition 9** *Let* $H = Span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ . *Then the number of vectors of a basis is called dimension of* $H$, *and is denoted as* $\dim H$.

**Definition 10** *(a) Let* $A = [\vec{a}_1, \vec{a}_2, ..., \vec{a}_p]_{m \times p}$ *be matrix that has* $p$ *column vectors* $\vec{a}_1, \vec{a}_2, ..., \vec{a}_p$. *We call the space spanned by columns*

$$Col(A) = Span\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_p\}$$

*the column space.*

*(b) The null space $Null\,(A)$ of $A$ is a subspace of all solution $A\vec{x} = \vec{0}$ :*

$$Null\,(A) = \left\{\vec{x} \mid A\vec{x} = \vec{0}\right\}.$$

*(c) If a subspace $H$ is spanned by $p$ vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_p$, i.e.,*

$$H = Span\,\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}\,,$$

*then we said that $H$ is generated by $\vec{v}_1, \vec{v}_2, ..., \vec{v}_p$, and these $\vec{v}_1, \vec{v}_2, ..., \vec{v}_p$ are called a set of generators for $H$. If in addition, generators $\vec{v}_1, \vec{v}_2, ..., \vec{v}_p$ are the linearly independent, then we call the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ a basis for $H$. The number of vectors in basis is called the dimension of $H$. In particular,*

$$\dim\left(Null\,(A)\right) = \textit{number of free variables.}$$

**Example 11** *Find a basis for (a) Null($A$) and (b) $Col\,(A)$ if*

$$A = \begin{bmatrix} -3 & 6 & -1 & -1 & 1 & -7 \\ 1 & -2 & 2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 5 & 8 & -4 \end{bmatrix}.$$

**Solution: (a)** We need to solve $A\vec{x} = \vec{0}$. We first perform row operations:

$$\begin{aligned} A &= \begin{bmatrix} -3 & 6 & -1 & -1 & 1 & -7 \\ 1 & -2 & 2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{bmatrix} 1 & -2 & 2 & 2 & 3 & -1 \\ -3 & 6 & -1 & -1 & 1 & -7 \\ 2 & -4 & 5 & 5 & 8 & -4 \end{bmatrix} \\ &\xrightarrow{\substack{R_2 + 3R_1 \to R_2 \\ R_3 - 2R_1 \to R_3}} \begin{bmatrix} 1 & -2 & 2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 5 & 10 & -10 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{bmatrix} \xrightarrow{\substack{R_2 - 5R_3 \to R_2 \\ R_1 - 2R_3 \to R_1}} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{bmatrix} \\ &\xrightarrow{R_2 - 5R_3 \to R_2} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The corresponding system equivalent to $A\vec{x} = \vec{0}$ becomes

$$\begin{aligned} x_1 - 2x_2 + x_5 + x_6 &= 0 \\ x_3 + x_4 + 2x_5 - 2x_6 &= 0. \end{aligned}$$

Since #2, #4, #5, #6 are non-pivot columns, $x_2$, $x_{4,}x_5, x_6$ are free variables, and they can be solved as

$$\begin{aligned} x_1 &= 2x_2 - x_5 - x_6 \\ x_3 &= -x_4 - 2x_5 + 2x_6, \end{aligned}$$

and solution has the parametric form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_5 - x_6 \\ x_2 \\ -x_4 - 2x_5 + 2x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

These 4 vectors are linearly independent and form a basis, and

$$Null\,(A) = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and $\dim (Null\,(A)) = 4$.

(b) From (a)

$$A = \begin{bmatrix} -3 & 6 & -1 & -1 & 1 & -7 \\ 1 & -2 & 2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 5 & 8 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Since in $B$, $\vec{b}_1$ and $\vec{b}_3$, the pivot columns of $B$, are linearly independent, and since $\vec{b}_2, \vec{b}_4, \vec{b}_5, \vec{b}_6$ are all linear combination of $\vec{b}_1$ and $\vec{b}_3$, we see that $\vec{b}_1$ and $\vec{b}_3$ form a basis and

$$Col\,(B) = Span \left\{ \vec{b}_1, \vec{b}_3 \right\}.$$

Note that row operations produce equivalent systems (i.e., systems have the same solutions), row operations will not change linear relations. More precisely, if $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 + x_5\vec{a}_5 + x_6\vec{a}_6 = \vec{0}$, then $x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3 + x_4\vec{b}_4 + x_5\vec{b}_5 + x_6\vec{b}_6 = \vec{0}$. We hence conclude

(a) $\vec{a}_1$ and $\vec{a}_3$ are linearly independent, since $\vec{b}_1$ and $\vec{b}_3$ are linearly independent, and

(b) $\vec{a}_2, \vec{a}_4, \vec{a}_5, \vec{a}_6$ are linear combination of $\vec{a}_1$ and $\vec{a}_3$ (e.g. $\vec{a}_2 = 2\vec{a}_1 + 3\vec{a}_3$), since $\vec{b}_2, \vec{b}_4, \vec{b}_5, \vec{b}_6$ are linear combination of $\vec{b}_1$ and $\vec{b}_3$ (e.g., $\vec{b}_2 = 2\vec{b}_1 + 3\vec{b}_3$).

and consequently

$$Col\,(A) = Span \{\vec{a}_1, \vec{a}_3\}$$

and

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \text{ form a basis.}$$

**Summary. Basis of $Col\,(A)$ = subset of pivot columns of $A$.Basis of $Null\,(A)$ =** vectors associated with free variables in a parametric vector form.

**Example 12** *Suppose that*

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 1 \\ 2 & -4 & 1 & 1 & 4 & 0 \\ -1 & 2 & 0 & 1 & 9 & 19 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 10 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$Then\ Col\,(A) = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} ???$$

*Correct answer*

$$Col\,(A) = Span \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

# Section 4.4 Coordinate Systems

Suppose that $\vec{v}_1, \vec{v}_{2,}..., \vec{v}_p$ form a basis for $H$. Then, for any $\vec{x} \in H$, we have the unique linear relation

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_p\vec{v}_p. \tag{1}$$

In fact, if there is another expression for the same $\vec{x}$ :

$$\vec{x} = d_1\vec{v}_1 + d_2\vec{v}_2 + ... + d_p\vec{v}_p,$$

then

$$c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_p\vec{v}_p = \vec{x} = d_1\vec{v}_1 + d_2\vec{v}_2 + ... + d_p\vec{v}_p,$$

and thus

$$(c_1 - d_1)\,\vec{v}_1 + (c_2 - d_2)\,\vec{v}_2 + ... + (c_p - d_p)\,\vec{v}_p = 0.$$

Since $\vec{v}_1, \vec{v}_{2,}..., \vec{v}_p$ are linearly independent,

$$c_1 - d_1 = 0, (c_2 - d_2) = 0, ..., c_p - d_p = 0,$$
$$\Longrightarrow$$
$$c_1 = d_1, c_2 = d_2, ..., c_p = d_p$$

i.e., the relation is unique.

**Definition 13** *Let* $B = \left\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\right\}$ *be a basis for* $H$*. Then for any vector* $\vec{x}$ *in* $H$*, there is a unique representation*

$$\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + ... + x_p\vec{b}_p.$$

*We call the vector*

$$[\vec{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

*the coordinate of* $\vec{x}$ *relative to the basis* $B$, *or* $B - coordinate$ *of* $\vec{x}$.

Example: Let B=$\left\{\vec{b}_1, \vec{b}_2\right\}$,

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ [\vec{x}]_B = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \vec{v}.$$

Find (a) $\vec{x}$ and (b) $[\vec{v}]_B$

Sol: (a) By the definition,

$$\vec{x} = -2\vec{b}_1 + 4\vec{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

(b) To find $B - coordinate$ of $\vec{v}$, we need to solve

$$v_1\vec{b}_1 + v_2\vec{b}_2 = \vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

The solution is

$$[\vec{v}]_B = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

**Definition 14** *Let* $B = \left\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\right\}$ *be a basis of a subspace* $H = Span\left\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\right\}$ *in* $R^n$. *Then*

$$P_B = \left[\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\right]$$

*is called the change of coordinate matrix from* $B$ *to the standard basis, and for any* $\vec{x}$ *in* $H$

$$\vec{x} = P_B\,[\vec{x}]_B = \left[\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\right][\vec{x}]_B$$

In the Example above,

$$P_B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},\quad P_B\,[\vec{v}]_B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \vec{v}$$

**Theorem 15** *Let $B_1$ and $B_2$ be two bases of $H$ in $R^n$. Then for any $\vec{x}$ in $H$,*

$$\vec{x} = P_{B_1} [\vec{x}]_{B_1} = P_{B_2} [\vec{x}]_{B_2}$$

*So if $H = R^n$, then*

$$[\vec{x}]_{B_1} = (P_{B_1})^{-1} P_{B_2} [\vec{x}]_{B_2}.$$

*The matrix $(P_{B_1})^{-1} P_{B_2}$ is called the change of coordinate matrix from $B_2$ to $B_1$.*

**Example 16** *Let*

$$\vec{b}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

*(a) Find a basis $B$ and dimension of $H = Span\left\{\vec{b}_1, \vec{b}_2\right\}$*
*(b) Determine if $\vec{x} \in H$. If so, find $[\vec{x}]_B$*

Sol: (a) Since $\vec{b}_1$ is not a multiple of $\vec{b}_2$, they are linearly independent. So $B = \left\{\vec{b}_1, \vec{b}_2\right\}$ forms a basis and $\dim H = 2$
(b) The basis changing matrix is

$$P_B = \begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix}$$

So $\vec{x} \in H$ iff the following system is consistent:

$$P_B \vec{y} = \vec{x}$$

By row operation, the augmented matrix

$$[P_B, \vec{x}] = \begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\vec{x} \in H,$ *and*

$$[\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

**Theorem 17** *(Coordinate mapping) Let $H$ be a $n$-dimensional vector space. Then there is an 1-1 and onto linear transformation $T$ from $H$ to $R^n$. In this case, we say $H$ is equivalent to $R^n$.*

Proof: Let $B = \left\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\right\}$ be a basis such that $H = Span\left\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\right\}$. Define $T : H \to R^n$ by

$$T(\vec{x}) = [\vec{x}]_B$$

**Example 18** *Let* $B = \{1, t, t^2, t^3\}$ *be the standard basis of* $P_3$*. Find* $T : P_3 \to R^3$*.*

**Example 19** *Let* $B = \{1 + 2t^2,\ 4 + t + 5t^2,\ 3 + 2t\}$ *be a subset of* $P_2$*. (a) Show* $B$ *forms a basis for* $P_3$ *and* $H = SpanB = P$*. (b) Find coordinate changing matrix from* $B$ *to the standard basis* $S = \{1.t.t^2\}$*.*

Sol:

$$\left[1 + 2t^2\right]_S = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ \left[4 + t + 5t^2\right]_S = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix},\ [3 + 2t]_S = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

For any quadratic function $f = a_0 + a_1 t + a_2 t^2$, if

$$[f]_B = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}, \text{ i.e.,}$$
$$f = f_0\left(1 + 2t^2\right) + f_1\left(4 + t + 5t^2\right) + f_2(3 + 2t)$$

then

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [f]_S$$
$$= \left[f_0\left(1 + 2t^2\right) + f_1\left(4 + t + 5t^2\right) + f_2(3 + 2t)\right]_S$$
$$= \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

So

$$P_B = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}, \quad [f]_S = P_B [f]_B$$

# 1