Section 3.1: Introduction to Determinants

For 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc.$$

For 3×3 matrix A, $\det(A)$ is defined through the following "diagonal rule"

$$[A, A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & \vdots & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & \vdots & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Example 3.1.1. Find $\det(A)$ if

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}.$$

Solution: Use "diagonal rule"

$$\begin{vmatrix} 1 & -2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \\ 2 & 4 & 1 & 2 & 4 & 1 \end{vmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{vmatrix} = 1 + (-2)\,3 \cdot 2 + 0 - 0 - (-2) \cdot 0 - 1 \cdot 3 \cdot 4 = -23.$$

Definition. Let $A = [a_{ij}]_n$ be a $n \times n$ square matrix. The $(n-1) \times (n-1)$ square matrix, denoted by A_{ij}, obtained by deleting ith row and jth column is called (i, j) −minor (or submatrix).

Example 3.1.2.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}, \; A_{11} = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \; A_{13} = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}.$$

Let us now compute

$$\det(A_{11}) = 1 - 12 = -11, \; \det(A_{12}) = -6, \; \det(A_{13}) = -2,$$

and it happens that

$$\begin{aligned} & a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) \\ & = 1 \cdot (-11) - (-2)(-6) + 0 = -23 = \det(A). \end{aligned}$$

Let A_n be a $n \times n$ square matrix. We use induction argument to define sucessively $\det(A_n)$. When $n = 2$, determinant $\det(A_2)$ for any 2×2 matrix is defined above. Suppose now that the determinants for all $(n-1) \times (n-1)$ matrices A_{n-1} are well defined.

Definition. We call

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

(i, j) -cofactor of A, and define $\det(A)$ by (cofactor expansion along ith row), for any i,

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} = a_{i1} C_{i1} + a_{i2} C_{i2} + ... + a_{in} C_{in}.$$

For instance, if $i = 1$, i.e., we expend along the first row, since

$$C_{11} = (-1)^{1+1} \det(A_{11}) = \det(A_{11}), \ C_{12} = (-1)^{1+2} \det(A_{12}) = -\det(A_{12}), ...$$

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - ... + (-1)^{n+1} \det(A_{1n}).$$

Note from the example above, this definition is consistent with the previous definition for 3×3 matrices. Note also that the sign $(-1)^{i+j}$ in C_{ij} changes alternatively as indicated below:

$$\begin{bmatrix} (1,1)+ & (1,2)- & (1,3)+ & \cdots \\ (2,1)- & (2,2)+ & (2,3)- & \cdots \\ (3,1)+ & (3,2)- & (3,3)+ & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Example 3.1.3. We use the same matrix in the previous example:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}, \ A_{21} = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}, \ A_{22} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}, \ A_{31} = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}, \ A_{32} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \ A_{33} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

$$\det(A_{21}) = -2, \ \det(A_{22}) = 1, \ \det(A_{23}) = 8$$
$$\det(A_{31}) = -6, \ \det(A_{32}) = 3, \ \det(A_{33}) = 1.$$

We now expand the cofactors along the second row:

$$\begin{aligned} & a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} \\ & = -a_{21} \det(A_{21}) + a_{22} \det(A_{22}) - a_{23} \det(A_{23}) \\ & = -0 + 1 \cdot 1 - 3 \cdot 8 = -23 = \det(A), \end{aligned}$$

We can also expand the cofactors along the third row

$$\begin{aligned} & a_{31} C3 + a_{32} C_{32} + a_{33} C_{33} \\ & = a_{31} \det(A_{31}) - a_{22} \det(A_{32}) + a_{23} \det(A_{33}) \\ & = 2(-6) - 4(3) + 1(1) = -23 = \det(A). \end{aligned}$$

In general, we have the following cofactor-expansion rule for determinants.

Theorem. $\det(A)$ can be calculated through a cofactor expansion along any row or column:

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} \quad \text{(cofactor expansion along } i\text{th row)}$$
$$= \sum_{k=1}^{n} a_{kj} C_{kj} \quad \text{(cofactor expansion along } j\text{th column).}$$

Example 3.1.4. Computer determinants using co-factor expansions. (a)

$$\det \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \overset{\text{1st column}}{=} 3 \det \begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$
$$\overset{\text{1st column}}{=} 3 \cdot 2 \det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \overset{\text{3rd row}}{=} 3 \cdot 2 \left(-(-2) \right) \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = -12$$

(b)

$$\det \begin{bmatrix} 3 & -7 & 8 & 9 \\ 0 & 2 & -5 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 3 \det \begin{bmatrix} 2 & -5 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix} = 6 \det \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix} = 24$$

(c) Upper triangle matrices

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & a_{22} & a_{14} & \cdots \\ 0 & 0 & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & * & * & \cdots \\ 0 & a_{33} & * & \cdots \\ 0 & 0 & a_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = ... = a_{11} a_{22} a_{33} ... a_{nn}$$

(d) Lower triangle matrices

$$\det \begin{bmatrix} a_{11} & 0 & 0 & \cdots \\ a_{21} & a_{22} & 0 & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & 0 & 0 & \cdots \\ * & a_{33} & 0 & \cdots \\ * & * & a_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = ... = a_{11} a_{22} a_{33} ... a_{nn}$$

Section 3.2: Properties of Determinants

The elementary row operations may simplify the calculation of determinants.

1. Single row (or column) linearity. For any fixed i, $\det()$ is linear in ith row (or column):

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & a_{i3}+b_{i3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix}$$

$$= \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} + \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & b_{i3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{i1} & \lambda a_{i2} & \lambda a_{i3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} = \lambda \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix}$$

In particular,

$$\det(\lambda A) = \lambda^n \det(A):$$

$$\det\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots \\ \cdots & \cdots & \cdots \\ \lambda a_{n1} & a_{n2} & \cdots \end{bmatrix} = \lambda \det\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \lambda a_{21} & \lambda a_{22} & \cdots \\ \cdots & \cdots & \cdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots \end{bmatrix} = ... = \lambda^n \det\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots \end{bmatrix}$$

Example 3.2.1.

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \det\begin{bmatrix} 1+0 & 2 & 3 & 4 \\ 0+2 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} + \det\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= 8 - 2\det\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 8 - 2 \cdot 8 = -8.$$

2. $\det\left(A^T\right) = \det(A)$

3. If two rows (or two columns) are interchanged, then the determinant changes sign.

   Suppose $R_1 \to R_2$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} \overset{R_1 \to R_2}{\longrightarrow} \begin{bmatrix} a_{21} & a_{22} & a_{23} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} = B.$$

   Then, we expand $\det(A)$ along 1st row, $\det(B)$ along 2nd row:

$$\det(A) \overset{first\ \text{row}}{=} a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})...$$
$$\det(B) \overset{\sec ond\ \text{row}}{=} -a_{11}\det(A_{11}) + a_{12}\det(A_{12}) - a_{13}\det(A_{13})...$$

   we find

$$\det(B) = -\det(A)$$

4. If one row is a multiple of another row (one column is a multiple of another column), then $\det(A) = 0$.

   In deed, suppose $R_2 = \lambda R_1$, then by one-row linearity,

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} = \lambda \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} = \det\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix}.$$

   On the other hand, by (3), since $R_1 \to R_2$,

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} = -\det\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix}.$$

   Therefore

$$\det\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix} = -\det\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots \end{bmatrix}$$

   (i.e., $x = -x$ iff $x = 0$).

Theorem. Let B is the matrix obtained by performing one time row operation from A.

Theorem 1 *1. If the row operation is type (1): a multiple of one row is added to another row, then* $\det(B) = \det(A)$;

2. if he row operation is type (2): two rows are interchanged, then $\det(B) = -\det(A)$;

3. If the row operation is type (3): one row is multipled by a constant k, then $\det(B) = k\det(A)$.

Proof. (3) Suppose that a multiple of 2nd row is added to the 1st row:

$$\det(B) = \det\begin{bmatrix} a_{11}+\lambda a_{21} & a_{12}+\lambda a_{22} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \det\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} + \det\begin{bmatrix} \lambda a_{21} & \lambda a_{22} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \det(A)$$

■

Example 3.2.2. Find determinants of the following matrices:

$$(1)\begin{bmatrix} -2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 9 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix},\ (2)\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}.$$

Sol: (1) We use only type 1 row operations:

$$\det\begin{bmatrix} -2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 9 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \overset{\substack{R_1+2R_4\to R_1 \\ R_2+R_3\to R_2 \\ R_3+3R_4\to R_3}}{=} \det\begin{bmatrix} 0 & 0 & 6 & 20 \\ 0 & 0 & 6 & 8 \\ 0 & -3 & 1 & 16 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 0 & 6 & 20 \\ 0 & 6 & 8 \\ -3 & 1 & 16 \end{bmatrix} = -(-3)\det\begin{bmatrix} 6 & 20 \\ 6 & 8 \end{bmatrix} = 3(48-120) = -216$$

(2)

$$\det\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix} \overset{R_3+2R_1\to R_3}{=} \det\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0.$$

More properties:

5 $\det(AB) = \det(A)\det(B)$

(By Theorem, it is true if A is an elementary matrix since row operation is the same as left multiplied by an elementary matrix. In general, A can be reduced to I if A is invertible).

6 A is invertible iff $\det(A) \neq 0$, and $\det(A^{-1}) = \det(A)^{-1}$

Since $A(A^{-1}) = I,\ \det(A)\det(A^{-1}) = 1,\ \det(A^{-1}) = 1/\det(A)$.

Practical advise in computing determinants: Try to use row operations to reduce a matrix to as close to either upper triangle or lower triangle matrix. Use only type 1 row operation if possible. If you have to use type 2, make sure record how many times you ever used. Never use type 3 row operation.

An alternative definition of determinant. Let

$$(l_1, l_2, ..., l_n) \quad \text{be an rearrangement of } (1, 2, ..., n), \text{ i.e., } l_i \neq l_j \text{ if } i \neq j.$$

For instance, with $n = 3$, there are total $3! = 6$ such rearrangements (also called permutations) (l_1, l_2, l_3) :

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1).$$

When $n = 4$, there are $4! = 24$ different permutations:

$$\begin{array}{l}(1,2,3,4), (1,3,2,4), (2,1,3,4), (2,3,1,4), (3,1,2,4), (3,2,1,4)\\ (1,2,4,3), (1,4,2,3), (2,1,4,3), (2,4,1,3), (4,1,2,3), (4,2,1,3)\\ (1,3,4,2), (1,4,3,2), (3,1,4,2), (3,4,1,2), (4,1,3,2), (4,3,1,2)\\ (2,3,4,1), (2,4,3,1), (3,2,4,1), (3,4,2,1), (4,2,3,1), (4,3,2,1).\end{array}$$

In general, there are $n!$ different permutations.

We call a permutation $(l_1, l_2, ..., l_n)$ EVEN if it can be obtained from $(1, 2, ..., n)$ by of a series of permuting only two components a time (single permutation) for a total of even number times. Otherwise, it is called ODD. For instance,

$$\begin{array}{ll}(1,2,3) & (\text{ 0 time})\\ (2,3,1) & (\text{ 2 times}):\ (1,2,3) \to (2,1,3) \to (2,3,1)\\ (3,1,2) & (\text{ 2 times}):\ (1,2,3) \to (1,3,2) \to (3,1,2)\end{array}$$

are all even, and the rest 3 are odd. Note that, for instance, it may take more than two single permutation to make $(2,3,1)$ from $(1,2,3)$. But the numbers remain even.

Theorem (optional) Let $A = [a_{ij}]_{n\times n}$. Then

$$\det(A) = \sum_{l_i \neq l_j (\text{for } i\neq j)} (-1)^{\langle l_1, l_2, ..., l_n\rangle} a_{1,l_1} a_{2,l2} a_{3,l_3} \cdots a_{n,l_n},$$

where

$$\langle l_1, l_2, ..., l_n\rangle = \begin{cases} 0, & \text{if } (l_1, l_2, ..., l_n) \text{ an even permutation}\\ 1, & \text{if } (l_1, l_2, ..., l_n) \text{ an odd permutation}\end{cases}.$$

Section 3.3: Cramer's Rule

Let A be a $n \times n$ matrix and consider

$$A\vec{x} = \vec{b}.$$

Let $A_i\left(\vec{b}\right)$ be a matrix obtained from A by replacing ith column by $\vec{b}$:

$$A_i\left(\vec{b}\right) = [\vec{a}_1, ..., \vec{a}_{i-1}, \vec{b}, \vec{a}_{i+1}, ...\vec{a}_n]$$

For instance,

$$A_1\left(\vec{b}\right) = \begin{bmatrix} b_1 & a_{12} & a_{13} & \cdots \\ b_2 & a_{22} & a_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_n & a_{n2} & a_{n3} & \cdots \end{bmatrix}, \ A_2\left(\vec{b}\right) = \begin{bmatrix} a_{11} & b_1 & a_{13} & \cdots \\ a_{21} & b_2 & a_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & b_n & a_{n3} & \cdots \end{bmatrix}, ...$$

Now, set $I_i(\vec{x}) = [\vec{e}_1...\vec{x}...\vec{e}_n]$. Then $\det(I_i(\vec{x})) = x_i$, if we expand along ith row. For instance,

$$I_2(\vec{x}) = \begin{bmatrix} 1 & x_1 & 0 & 0 & \cdots \\ 0 & x_2 & 0 & 0 & \cdots \\ 0 & x_3 & 1 & 0 & \cdots \\ 0 & x_4 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \ \det(I_2(\vec{x})) \overset{\text{2nd row expansion}}{=} x_2 \det(I)$$

Since

$$\begin{aligned} AI_i(\vec{x}) &= A[\vec{e}_1...\vec{x}...\vec{e}_n] = [A\vec{e}_1...A\vec{x}...A\vec{e}_n] \\ &= [\vec{a}_1, ..., \vec{a}_{i-1}, \vec{b}, \vec{a}_{i+1}, ...\vec{a}_n] = A_i\left(\vec{b}\right), \end{aligned}$$

we find

$$\det(A)\det(I_i(\vec{x})) = \det\left(A_i\left(\vec{b}\right)\right), \quad \det(A)\, x_i = \det\left(A_i\left(\vec{b}\right)\right).$$

Theorem (Cramer's Rule) The solution of $A\vec{x} = \vec{b}$ is

$$x_i = \frac{\det\left(A_i\left(\vec{b}\right)\right)}{\det(A)}.$$

Example 3.3.1. Use Cramer's rule to solve

$$\begin{aligned} 3x_1 + 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8. \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 3 & 2 \\ -5 & 4 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$A_1\left(\vec{b}\right) = \begin{bmatrix} 6 & 2 \\ 8 & 4 \end{bmatrix}, \ A_2\left(\vec{b}\right) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

$$x_1 = \frac{\det\left(A_1\left(\vec{b}\right)\right)}{\det(A)} = \frac{24-16}{12-(-10)} = \frac{8}{22} = \frac{4}{11}$$

$$x_2 = \frac{\det\left(A_2\left(\vec{b}\right)\right)}{\det(A)} = \frac{24-(-30)}{22} = \frac{54}{22} = \frac{27}{11}.$$

We nex use Cramer's rule to calculate A^{-1}. Since

$$A^{-1} = A^{-1}I = A^{-1}[\vec{e}_1, ..., \vec{e}_n] = [A^{-1}\vec{e}_1, ..., A^{-1}\vec{e}_n],$$

it suffices to find $A^{-1}\vec{e}_i$. Obviously, $A^{-1}\vec{e}_i$ is the solution of $A\vec{x} = \vec{e}_i$, by Cramer's rule

$$A^{-1}\vec{e}_i = \frac{1}{\det(A)} \begin{bmatrix} \det A_1(\vec{e}_i) \\ \det A_2(\vec{e}_i) \\ \det A_3(\vec{e}_i) \\ \cdots \end{bmatrix}$$

where

$$A_1(\vec{e}_i) = [\vec{e}_i, \vec{a}_2, ...\vec{a}_n] = [\begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}, \vec{a}_2, ...\vec{a}_n].$$

If we expand along the first column, since only the ith element is not zero (actually $= 1$) in $\vec{e}_i$,

$$\det A_1(\vec{e}_i) = (-1)^{i+1} \det A_{i1} = C_{i1}$$

In general,

$$\det A_j(\vec{e}_i) = (-1)^{i+j} \det A_{ij} = C_{ij}.$$

Now

$$A^{-1}\vec{e}_i = \frac{1}{\det(A)} \begin{bmatrix} \det A_1(\vec{e}_i) \\ \det A_2(\vec{e}_i) \\ \det A_3(\vec{e}_i) \\ \cdots \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{i1} \\ C_{i2} \\ C_{i3} \\ \cdots \end{bmatrix},$$

$$A^{-1} = [A^{-1}\vec{e}_1, ..., A^{-1}\vec{e}_n] = \begin{bmatrix} \frac{C_{11}}{\det(A)} & \frac{C_{21}}{\det(A)} & \cdots \\ \frac{C_{12}}{\det(A)} & \frac{C_{22}}{\det(A)} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

Cramer's Rule for inverse matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} & \cdots \\ C_{12} & C_{22} & C_{32} & \cdots \\ C_{13} & C_{23} & C_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad ([C_{ji}] \text{ is called adjugate matrix})$$

where

$$C_{ij} = (-1)^{i+j} \det(A_{ij}) \quad \text{is } (i,i) - \text{cofactor of } A.$$

Example 3.3.2. Find A^{-1} using Cramer's Rule:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

Solution: We first computer "adjoint" matrix

$$[\det A_{ij}] = \begin{bmatrix} 2-4 & -2-1 & 4-(-1) \\ -2-12 & -4-3 & 8-1 \\ 1-(-3) & 2-3 & -2-1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 5 \\ -14 & -7 & 7 \\ 4 & -1 & -3 \end{bmatrix}$$

From this matrix, since $C_{ji} = (-1)^{i+j} \det A_{ij}$, we can easily find the adjugate matrix,

$$C = [C_{ji}] = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix},$$

and determinant

$$\det(A) \overset{\text{1st row expansion}}{=} (\text{1st row of } A) \cdot (\text{1st column of } C?)$$
$$= \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T \cdot \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = 14.$$

Therefore,

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}.$$