Section 2.3: Characterization of Invertible Matrices

We summarize the entire chapter 2 by the following theorem.

Theorem 1 *(Invertible matrix theorem I&II) Let $A_{n\times n}$ be a square matrix. The following ststements are equivalent:*

1. *The matrix A is invertible,*
2. *A^T is invertible,*
3. *$AB = AC \Longrightarrow B = C$ (i.e., cancellation law holds for A)*
4. *A is left-invertible, i.e., there exists C such that $CA = I$,*
5. *A is right-invertible, i.e., there exists D such that $AD = I$,*
6. *The reduced Echelon form of A is identity matrix I,*
7. *A is row-equivalent to identity matrix I,*
8. *A has n pivot positions,*
9. *$A\vec{x} = \vec{0}$ has only trivial solution $\vec{x} = \vec{0}$,*
10. *Column vectors of A are linearly independent,*
11. *Column vectors of A span R^n,*
12. *Row vectors of A span R^n,*
13. *$A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in R^n, //*
14. *The columns of A form a basis for $Col\,(A)$,*
15. *$Col\,(A) = R^n$,*
16. *$Rank(A) = n$*
17. *$Rank(Null\,(A)) = 0$.*

Proof. (Very briefly) (1) $\Longleftrightarrow$ (2) : because $\left(A^T\right)^{-1} = (A^{-1})^T$

(1) $\Longrightarrow$ (3) : $AB = AC \Longrightarrow A\,(B - C) = 0 \Longrightarrow A^{-1}A\,(B - C) = 0 \Longrightarrow (B - C) = 0$

(1) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (5) : obviously;

(1) $\Longrightarrow$ (5) : $E_p ... E_2E_1A = I \Longrightarrow E_p ... E_2E_1 = A^{-1}$

(6) $\Longleftrightarrow$ (7) : $A \sim I$

(7) $\Longleftrightarrow$ (8) : I has n pivots

(8) $\Longrightarrow$ (9) : no free variable

(9) $\Longrightarrow$ (10) : any linear realtion $\vec{x}$ is a solution of $A\vec{x} = \vec{0}$

(8) $\Longrightarrow$ (11) : $A\vec{x} = \vec{b}$ is always consistent since by (10), the last column in the augmented matrix is non-pivot

(2) $\Longrightarrow$ (12) : columns of A^T are exactly the rows of A

(11) $\Longleftrightarrow$ (13) : re-statements.

(11) $\Longleftrightarrow$ (15) : re-statements.

(16) $\Longleftrightarrow$ (17) : dimension theorem ■