Section 2.2: Inverse of a Matrix

Definition 1 *A square matrix $A_{n\times n}$ is said to be invertible if there exists a unique matrix $C_{n\times n}$ of the same size such that*

$$AC = CA = I_n.$$

The matrix C is called the inverse of A, and is denoted by

$$C = A^{-1}$$

Suppose now $A_{n\times n}$ is invertible and $C = A^{-1}$. The matrix equation $A\vec{x} = \vec{b}$ can be easily solved as follows

$$\begin{aligned} A\vec{x} = \vec{b} \Longrightarrow CA\vec{x} = C\vec{b} \Longrightarrow I_n\vec{x} = C\vec{b} \Longrightarrow \\ \vec{x} = A^{-1}\vec{b}. \end{aligned} \tag{1}$$

Example 2 *(a) Show A is invertible and $A^{-1} = C$, where*

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}, \quad C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}.$$

*(b) Show that the matrix A in Example (**??**) is NOT invertible:*

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution: (a) Direction calculations lead to $AC = CA = I_2$.

(b) From Example (**??**), we know that $A^2 = 0$. Now, suppose A is indeed invertible. Then there exists a matrix C such that

$$AC = I_2.$$

Multiplying by A from the left, we find

$$AAC = AI_2 \Longrightarrow A^2C = A \Longrightarrow 0 = A \Longrightarrow \text{contradiction.}$$

Theorem 3 *For 2×2 matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

it is invertible iff $\det(A) \overset{def}{=} ad - bc \neq 0$. When $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 4 *(a) Find A^{-1} if*

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -6 \end{bmatrix}.$$

(b) Solve

$$\begin{aligned} 2x + 5y &= 1 \\ -3x - 6y &= 2 \end{aligned}$$

Solution: (a) $a = 2, b = 5, c = -3, d = -6$. $ad - bc = -12 - (-15) = 3$, and

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -6 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -5/3 \\ 1 & 2/3 \end{bmatrix}.$$

We may verify the above solution as follows:

$$\begin{bmatrix} 2 & 5 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} -2 & -5/3 \\ 1 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) To solve the system, we write its matrix equation: $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -6 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

According Equ. (1), the solution is

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -2 & -5/3 \\ 1 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{16}{3} \\ \frac{7}{3} \end{bmatrix}.$$

Properties if Invertible Matrix:

Suppose that A is an invertible square matrix. Then

Claim 5 *1. A^{-1} is also invertible and $(A^{-1})^{-1} = A$.*

2. A^T is also invertible and $\left(A^T\right)^{-1} = (A^{-1})^T$.

3. If B is another invertible matrix of the same size, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

4. $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.

5. The reduced Echelon of A is the identity matrix I of the same size, i.e., $A \longrightarrow I$.

Proof. (1) By definition, A^{-1} if we can find a matrix C such that

$$\left(A^{-1}\right)C = C\left(A^{-1}\right) = I.$$

The above is indeed true if $C = A$.

(2) Take transposes of all three sides of $\left(A^{-1}\right)A = A\left(A^{-1}\right) = I$,

$$\left(\left(A^{-1}\right)A\right)^T = \left(A\left(A^{-1}\right)\right)^T = I^T \implies A^T\left(A^{-1}\right)^T = \left(A^{-1}\right)^T A^T = I$$
$$\implies A^T C = CA^T = I, \quad \text{where } C = \left(A^{-1}\right)^T \text{ is the inverse of } A^T.$$

(3) Let $C = B^{-1}A^{-1}$. Then this C is the inverse of AB since

$$C(AB) = (CA)B = \left(\left(B^{-1}A^{-1}\right)A\right)B = \left(B^{-1}\left(A^{-1}A\right)\right)B = \left(B^{-1}I\right)B = \left(B^{-1}\right)B = I$$
$$(AB)C = A(BC) = A\left(B\left(B^{-1}A^{-1}\right)\right) = A\left(\left(BB^{-1}\right)A^{-1}\right) = A\left(IA^{-1}\right) = AA^{-1} = I.$$

(5) Since $A\vec{x} = 0$ has a solution, there is no non-trivial solution, and thus A has no non-pivot column. ■

Definition 6 *A matrix is called an elementary matrix if it is obtained from performing one single elementary row operation on an identity matrix.*

Example 7 *Let us look at 3×3 elementary matrices by corresponding row operations. Type (1) elementary matrix E_1 :*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + \lambda R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + (-2)R_1 \to R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$
$$\iff \textit{left multiplication by } E_1 \textit{ with } \lambda = -2 \text{ , } \textit{i.e.,}$$

$$E_1 \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Type (1) elementary row operation = left multiplicationby by a Type (1) elementary matrix E_1

Type (2):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_1} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 4 & 0 & 1 \end{bmatrix}$$

$\Longleftrightarrow$ *left multiplication by* E_2 :

$$E_2 \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 4 & 0 & 1 \end{bmatrix}$$

Type (2) elementary row operation = left multiplicationby by a Type (2) elementary matrix E_2

Type (3):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\lambda R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} = E_3$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{4} R_3 \to R_3} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1/4 \end{bmatrix}$$

$\Longleftrightarrow$ *left multiplication by* E_3 *with* $\lambda = 1/4$:

$$E_3 \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1/4 \end{bmatrix}.$$

Type (3) elementary row operation = left multiplicationby by a Type (3) elementary matrix E_3

Conclusion 8 *Performing an elementary row operation on an identity matrix produces an elementary matrix corresponging that elementary row operation. Any elementary row operation = left multiply by the corresponding elementary matrix.*

Suppose now A is invertible. Then, by a series of successive row operations, the matrix A is reduced to I. Since each row operation is equivalent to left multiplication by an elementary matrix, this means that there exist elementary matrices E_1, $E_2, ..., E_k$, such that

$$(E_k ... E_2 E_1) A = I \implies A^{-1} = E_k ... E_2 E_1 = E_k ... E_2 E_1 (I).$$

The very last equation says that the exact same row operations that reduce A to the identity I, in the same time, also reduce the identity I to A^{-1}.

Inverse matrix Algorithm:

$$[A, I] \xrightarrow{\text{elementary row operations}} \left[I, A^{-1}\right]$$

Example 9 *Find* A^{-1} *if*

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -2 & 0 & 15 \end{bmatrix}.$$

Solution:

$$[A, I] = \begin{bmatrix} 1 & 3 & -1 & 1 & 0 & 0 \\ 2 & 7 & 0 & 0 & 1 & 0 \\ -2 & 0 & 15 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 6 & 13 & 2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 - 6R_2 \to R_3]{R_1 - 3R_2 \to R_1} \begin{bmatrix} 1 & 0 & -7 & 7 & -3 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 14 & -6 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - 2R_3 \to R_2]{R_1 + 7R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 105 & -45 & 7 \\ 0 & 1 & 0 & -30 & 13 & -2 \\ 0 & 0 & 1 & 14 & -6 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 105 & -45 & 7 \\ -30 & 13 & -2 \\ 14 & -6 & 1 \end{bmatrix}$$

Verify:

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -2 & 0 & 15 \end{bmatrix} \begin{bmatrix} 105 & -45 & 7 \\ -30 & 13 & -2 \\ 14 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$