Section 2.1. Matrix Operations

0.1 Matrix additions and Scalar Multiplications

Definition 1 *Let* $A = [\vec{a}_1, \vec{a}_2, ..., \vec{a}_n]_{m\times n} = [a_{ij}]_{m\times n}$ *and* $B = \left[\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\right]_{m\times n} = [b_{ij}]_{m\times n}$ *be two matrices of the same dimension, where columns* $\vec{a}_i$ *and* $\vec{b}_j \in R^m$, *and* λ *be a constant (scalar). Then*

$$\begin{aligned} A + B &= \left[\vec{a}_1 + \vec{b}_1, \vec{a}_2 + \vec{b}_2, ..., \vec{a}_n + \vec{b}_n\right]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n} \\ \lambda A &= [\lambda\vec{a}_1, \lambda\vec{a}_2, ..., \lambda\vec{a}_n]_{m\times n} = [\lambda a_{ij}]_{m\times n}. \end{aligned}$$

Or in the expanded form,

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & ... & \vdots \\ a_{m1} & ... & ... & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & a_{2n} \\ \vdots & \vdots & ... & \vdots \\ b_{m1} & ... & ... & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & ... & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & ... & \vdots \\ \vdots & \vdots & ... & \vdots \\ a_{m1} + b_{m1} & ... & ... & a_{mn} + b_{mn} \end{bmatrix},$$

$$\lambda \begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & ... & \vdots \\ a_{m1} & ... & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & ... & \lambda a_{1n} \\ \vdots & ... & \vdots \\ \lambda a_{m1} & ... & \lambda a_{mn} \end{bmatrix}.$$

Example 2 *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 10 \end{bmatrix}, \ 3A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ A - B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 8 \end{bmatrix}, \quad A + C = ?.$$

Properties:

$$\begin{aligned} A + B &= B + A \\ (A + B) + C &= A + (B + C) \\ \lambda(A + B) &= \lambda A + \lambda B. \end{aligned}$$

0.2 Matrix Multiplications

Multiplication between two matrices is defined as follows.

Definition 3 *Let* $A = [\vec{a}_1, \vec{a}_2, ..., \vec{a}_n]_{m\times p} = [a_{ij}]_{m\times p}$ *and* $B = \left[\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\right]_{p\times n} = [b_{ij}]_{p\times n}$ *be two matrices, where columns*

$$\vec{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} \in R^m, \ i = 1, 2, ..., p, \quad \vec{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} \in R^p,$$

respectively. Note that

$$\textit{number of columns in } A = \textit{ number of rows in } B = p.$$

Then the product of A *and* B *is*

$$AB = \left[A\vec{b}_1, A\vec{b}_2, ..., A\vec{b}_n\right]_{m\times n} = [c_{ij}]_{m\times n},$$

with

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Another way to look at this definition of multiplication is to the dot product. Rewrite

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1p} \\ a_{21} & a_{22} & ... & a_{2p} \\ \vdots & \vdots & ... & \vdots \\ a_{m1} & ... & ... & a_{mp} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix},$$

where $n \times 1$ matrix

$$A_i = \begin{bmatrix} a_{i1} & a_{i2} & ... & a_{ip} \end{bmatrix} \text{ is the } ith \text{ row of } A.$$

is called a row vector. Without confusion, we can view a row vector as an equivalent to a column vector as follows:

$$\begin{bmatrix} a_{i1} & a_{i2} & ... & a_{ip} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{ip} \end{bmatrix}.$$

Then, we have the following so-called "row-column rule"

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \left[\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\right]$$

$$= \begin{bmatrix} A_1b_1 & A_1b_2 & \cdots & A_1b_n \\ A_2b_1 & A_2b_2 & \cdots & A_2b_n \\ \vdots & \vdots & A_ib_j \text{ in row } i \text{ column } j & \vdots \\ A_mb_1 & A_mb_2 & \cdots & A_mb_n \end{bmatrix} = [A_ib_j]_{mn},$$

$$A_ib_j = c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

Example 4 *Determine whether the product AB or BA makes sense if*

$$(a) \quad A = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$(b) \quad A = \begin{bmatrix} 3 & 5 & 7 & 1 \\ 9 & 11 & 13 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$(c) \quad A = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}.$$

$$(d) \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Find AB or BA if it makes sense. What is the size of AB, BA? Is AB equal to BA if both make sense?

Solution: (a) Neither –wrong sizes: $A_{2\times3}$, $B_{2\times3}$. (b) only BA makes sense – $A_{2\times4}$, $B_{3\times2}$. (c) Both make sense– $A_{2\times3}$, $B_{3\times2}$, and

$$AB = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} = \left[A \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, A \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 3\cdot1+5\cdot4+7\cdot7 & 3\cdot2+5\cdot5+7\cdot8 \\ 9\cdot1+11\cdot4+13\cdot7 & 9\cdot2+11\cdot5+13\cdot8 \end{bmatrix} = \begin{bmatrix} 72 & 87 \\ 144 & 177 \end{bmatrix}_{2\times2}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 33 \\ 57 & 75 & 93 \\ 93 & 123 & 153 \end{bmatrix}_{3\times3}.$$

Note that AB and BA have different dimensions. (d) Both – $A_{2\times 2}$, $B_{2\times 2}$, and

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ 3 & 0 \end{bmatrix}_{2\times 2}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 15 & 0 \end{bmatrix}_{2\times 2}$$

Apparently, even if AB and BA have the same size 2×2,

$$AB \neq BA.$$

Definition 5 *Let $A = [a_{ij}]_{m\times p}$. The transpose of A, denoting by A^T, is a matrix of $p \times m$ whose ith row is the ith column of the original matrix A (with the equivalence described above). In other words, A^T is formed by swapping rows and columns of A.*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{bmatrix}_{m\times p}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1p} & a_{2p} & \dots & a_{mp} \end{bmatrix}_{p\times m}$$

In particular,

$$\begin{bmatrix} a_1 & a_2 & \dots & a_p \end{bmatrix}^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}.$$

Sometimes we simply treat a row vector as if it is a column vector, i.e.,

$$\begin{bmatrix} a_1 & a_2 & \dots & a_p \end{bmatrix} \Leftrightarrow \begin{bmatrix} a_1 & a_2 & \dots & a_p \end{bmatrix}^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}.$$

Definition 6 *If $m = p$, then the kth power can be defined and is defined as*

$$A^2 = AA, \quad A^3 = A^2A, \dots, A^k = A^{k-1}A.$$

(If A_{mp} is not a square matrix, then $A_{mp}A_{mp} =$?.)

Example 7

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & 9 \\ 5 & 11 \\ 7 & 13 \end{bmatrix};$$

$$B = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 0 & 2 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 3 & 9 & 0 \\ 5 & 11 & 2 \\ 7 & 13 & 0 \end{bmatrix}.$$

If we denote A_i the ith row in A, i.e.,

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, \quad A_i = \begin{bmatrix} a_{i1} & a_{i2} & ... & a_{ip} \end{bmatrix}, \; A_i^T = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{ip} \end{bmatrix}$$

then $(A_i)^T$ is a column vectors in R^p. The "row-column rule" becomes

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}_{m\times p} \begin{bmatrix} \vec{b}_1, \vec{b}_2, ..., \vec{b}_n \end{bmatrix}_{p\times n} = \begin{bmatrix} (A_1)^T \cdot \vec{b}_1 & (A_1)^T \cdot \vec{b}_2 & ... & (A_1)^T \cdot \vec{b}_n \\ (A_2)^T \cdot \vec{b}_1 & (A_2)^T \cdot \vec{b}_2 & ... & (A_2)^T \cdot \vec{b}_n \\ ... & ... & ... & ... \\ (A_m)^T \cdot \vec{b}_1 & (A_m)^T \cdot \vec{b}_2 & ... & (A_m)^T \cdot \vec{b}_n \end{bmatrix}_{m\times n} = \begin{bmatrix} (A_i)^T \cdot \vec{b}_j \end{bmatrix}_{m\times n}$$

Example 8 *Use the row-column rule:*

$$\begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 5 & -1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ -3 & -10 & 11 \\ -1 & 0 & -3 \end{bmatrix}.$$

0.3 Properties:

Let A, B, C are matrices whose sizes vary and will be indicated by subscripts, λ be a constant, and I_m be $m \times m$ identity matrix. Then

1. $A(BC) = (AB)C \iff A_{m\times p}(B_{p\times n}C_{n\times r})_{p\times r} = (A_{m\times p}B_{p\times n})_{m\times n}C_{n\times r}$
2. $A(B+C) = AB + AC \iff A_{m\times p}(B_{p\times n} + C_{p\times n}) = A_{m\times p}B_{p\times n} + A_{m\times p}C_{p\times n}$
3. $(B+C)A = BA + CA \iff (B_{p\times n} + C_{p\times n})A_{n\times m} = B_{p\times n}A_{n\times m} + C_{p\times n}A_{n\times m}$
4. $\lambda(AB) = (\lambda A)B = A(\lambda B) \iff \lambda(A_{m\times p}B_{p\times n}) = (\lambda A_{m\times p})B_{p\times n} = A_{m\times p}(\lambda B_{p\times n})$
5. $IA = AI = A \iff I_m A_{m\times p} = A_{m\times p} I_p = A_{m\times p}$
6. $(\lambda A)^T = \lambda A^T$
7. $(A^T)^T = A$
8. $(B+C)^T = B^T + C^T \iff (B_{p\times n} + C_{p\times n})^T = (B_{p\times n})^T + (C_{p\times n})^T$
9. $(BC)^T = C^T B^T \iff (B_{p\times n}C_{n\times r})^T = (C_{n\times r})^T (B_{p\times n})^T$

where

$$I_m = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}_{m \times m} \quad \text{is called identity matrix of dimension } m.$$

Example 9 *(a) Let*

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \textit{(note: } A \neq 0, \textit{ no "cancellation rule")} \\ AB &= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ BA &= \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad \textit{(note: } AB \neq BA) \\ B^T A^T &= \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = (AB)^T. \end{aligned}$$