Section 1.9: The Matrix of a Linear Transformation

For any $m \times n$ matrix A, one can define a linear transformation T from R^n to R^m as follows: for any $\vec{u} \in R^n$

$$T(\vec{u}) = (A_{m\times n})(\vec{u}_{n\times 1}).$$

In particular, if we write $A = [\vec{a}_1\ \vec{a}_2\ ...\ \vec{a}_n]$, where $\vec{a}_i$ is the ith column vector of A, then

$$T(\vec{e}_i) = A\vec{e}_i = \vec{a}_i$$

This motives us to define the matrix for any linear transformation T from R^n to R^m.

Definition: For any linear transformation T from R^n to R^m, the $m \times n$ matrix A defined above, i.e.,

$$A = [T(\vec{e}_1)\ \ T(\vec{e}_2)\ \ ...\ T(\vec{e}_n)]_{mxn}$$

is called the matrix of T under the standard basis.

Proposition: Any linear transformation T from R^n to R^m can be representated by its matrix A_T in the following sense: For any column vector $\vec{x}$ in R^n,

$$T(\vec{x}) = A_T\vec{x},\ \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Proof: We can write $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ ...x_n\vec{e}_n.\ So$

$$T(\vec{x}) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \ ...x_nT(\vec{e}_n) = [T(\vec{e}_1)\ \ T(\vec{e}_2)\ \ ...\ T(\vec{e}_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A_T\vec{x}$$

Property: (1) Linearity: Let T and S are two linear transformations from R^n to R^m, then for any constants α and β, the matrix for $\alpha T + \beta S$ is $\alpha A_T + \beta A_S,\ i.e.,$

$$A_{\alpha T+\beta S} = \alpha A_T + \beta A_S$$

(2)Let T and S are two linear transformations from R^n to R^m, and from R^m to R^r, respectively. Then the matrix for $S \circ T$ is $A_S A_T$, $i.e.,$

$$A_{S\circ T} = A_S A_T.$$

Example