Section 1.8: Linear Transformations

Definition 1 *A linear transformation is a mapping (or function) T from R^n to R^m satisfying (i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and (ii) $T(\lambda\vec{u}) = \lambda T(\vec{u})$ for any real number λ.*

Example 2 *In 1-D, $T(x) = cx$ (c is a constant) is a linear transformation. But $T(x) = ax + b$ is NOT (called affine transformation).*

Any $m \times n$ matrix $A_{m\times n}$ defines a linear transformation T from R^n to R^m as follows: for any $\vec{u} \in R^n$

$$T(\vec{u}) = (A_{m\times n})(\vec{u}_{n\times 1}). \tag{1}$$

We can show that for any linear transformation $T$from R^n to R^m, there is a $m \times n$ matrix $A_{m\times n}$ such that (1) holds. In other words, any linear transformation can be defined by a matrix.

Example 3 *(a) $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ is called a dilation if $r > 1$ and is contraction if $0 < r < 1$. For any $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$,*

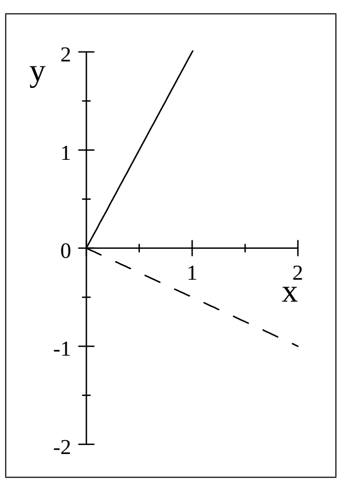
$$A\vec{u} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} x \\ y \end{bmatrix} = r\vec{u}.$$

(b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is called a rotation (rotation counter-clockwisely by $\frac{\pi}{2}$):

$$A\vec{u} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix},$$

$$(A\vec{u}) \cdot \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = 0.$$

For instance, $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

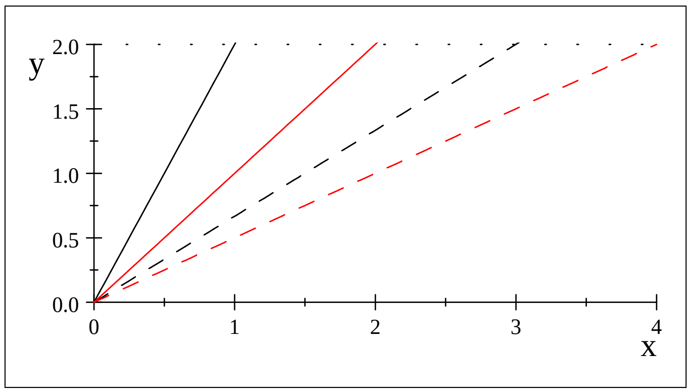


(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *is a shear:*

$$A\vec{u} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} = \vec{u} + \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

For instance,

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$



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