1 Section 1.7. Linear Independence

Definition 1 *A set of p vectors* $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ *in* R^m *is called linearly independent if the vector equation*

$$x_1\vec{u}_1 + x_2\vec{u}_2 + ... + x_p\vec{u}_p = \vec{0}$$

has only the trivial solution $x_i = 0$, $i = 1, 2, ..., p$. *Otherwise, the set is called linearly dependent; the coefficients* $x_1, ..., x_n$ *are called a linear relation.*

Example 2 *Any one single vector* $\vec{u}$ *is always independent. two vectors* $\vec{u}_1, \vec{u}_2$ *are dependent iff* $\vec{u}_1 = \lambda\vec{u}_2$.

The notation of linear dependence (or independence) is closely related to homogeneous systems. In fact, let $A = [\vec{a}_1, \vec{a}_2, ..., \vec{a}_n]$ be a $m \times n$ matrix with column vectors $\vec{a}_i$, then we have the following relation:

Claim 3 $A\vec{x} = \vec{0}$ *has NO non-trivial solution iff its column vectors* $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ *are linearly independent.*

Example 4 *(1) Determine whether the set of the following three vectors is dependent:*

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

(2) Find a linear relation.

Solution. (1) We need to determine if $A\vec{x} = \vec{0}$ has a non-trivial solution, where $A = [\vec{u}_1, \vec{u}_2, \vec{u}_3]$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 0 & 6 \end{bmatrix}.$$

To this end, we perform row operation on the augmented matrix $\left[A, \vec{0}\right]$.For simplicity, we only work on A :

$$A \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \text{ since } \left[A, \vec{0}\right] \to \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (1)$$

Apparently, x_3 is a free variable. Thus, $A\vec{x} = \vec{0}$ has non-trivial solution, and the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly dependent.

Solution. (2) Finding a linear relation $\vec{u}_1, \vec{u}_2, \vec{u}_3$, i.e.,

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = \vec{0}$$

means to find a non-trivial solution of

$$A\vec{x} = \vec{0}.$$

From (1), we find that $A\vec{x} = \vec{0}$ reduces to

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ -3x_2 - x_3 &= 0. \end{aligned}$$

Since we only need one non-zero solution, we take $x_3 = 3$ (we could have choose any number here. The only reason I chose 3 is to avoid fraction.) Then $x_2 = -1$, $x_1 = -(2x_2 + 4x_3) = -10$, and

$$-10\vec{u}_1 - \vec{u}_2 + 3\vec{u}_3 = \vec{0}.$$

From the above equation, we see that, for instance,

$$\vec{u}_2 = -10\vec{u}_1 + 3\vec{u}_3,$$

i.e., when $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly dependent, $\vec{u}_2$ is a linear combination of $\{\vec{u}_1, \vec{u}_3\}$. In general,

Theorem 5 *(Characterization of linear dependence) Vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ are linearly dependent iff at least one is a linear combination of the rest.*

Proof. Suppose that one is a linear combination of the rest. Without loss of generality, we say that $\vec{u}_1$ is a linear combination of $\vec{u}_2, ..., \vec{u}_p$, i.e., there exist $\lambda_2, ..., \lambda_p$ such that

$$\vec{u}_1 = \lambda_2\vec{u}_2 + ... + \lambda_p\vec{u}_p.$$

It follows that

$$-\vec{u}_1 + \lambda_2\vec{u}_2 + ... + \lambda_p\vec{u}_p = 0.$$

This implies that $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ are linearly dependent. On the other hand, suppose that $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ are linearly dependent. Then, we have

$$\lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + ... + \lambda_p\vec{u}_p = 0,$$

where at least one of p constants $\{\lambda_1, \lambda_2, ..., \lambda_p\}$ is not zero. Without loss of generality, we say $\lambda_1 \neq 0$. Then we can solve for $\vec{u}_1$ as

$$\vec{u}_1 = \left(\frac{\lambda_2}{\lambda_1}\right)\vec{u}_2 + ... + \left(\frac{\lambda_p}{\lambda_1}\right)\vec{u}_p.$$

Hence, $\vec{u}_1$ is a linear combination of $\vec{u}_2, ..., \vec{u}_p$, the rest. ■

Note that when $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ are linearly dependent, it does NOT mean ANY one member is a linear combination of the rest. For instance,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{are linearly dependent.}$$

But

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is not a linear combination of } \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

In summary, to determine whether $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ in R^m are linearly independent, we need to see whether the system $A\vec{x} = \vec{0}$ has only trivial solution, where A is the $m \times p$ column matrix $[\vec{u}_1, \vec{u}_2, ..., \vec{u}_p]$.

Definition 6 *We call the number of pivots of a $m \times p$ matrix A the RANK of A, and denote it by $r(A)$.*

Obviously, the rank cannot exceed the number of rows or columns, i.e., $r(A) \leq m$, $r(A) \leq p$.

The following theorem follows directly from the last conclusion in Section 1.5.

Theorem 7 *A set of vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ in R^m is linearly dependent iff $p > r(A)$, where A is the $m \times p$ column matrix $[\vec{u}_1, \vec{u}_2, ..., \vec{u}_p]$. In particular, it is linearly dependent if $p > m$, i.e., the number of vectors is more than the dimension.*

Example 8 *Determine whether the following set is linearly dependent. If it is linearly dependent find a set of linearly independent vectors $\vec{v}_1, \vec{v}_2, ...$ such that $Span\{\vec{v}_1, \vec{v}_2, ..\} = Span\{\vec{u}_1, \vec{u}_2, ..\}$.*

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. The answer is yes, it is linearly dependent because the number of vectors is great than the dimension.

We next describe all vectors $\vec{b} \in Span\{\vec{u}_1, \vec{u}_2, ..\}$ using parametric vector representation. As we did in Example **??**, we need to describe $\vec{b}$ such that $[\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4]\ \vec{x} = \vec{b}$ has a solution. To this end, we perform row operation on $\left[\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{b}\right]$ to arrive at a Echelon form:

$$\begin{bmatrix} 1 & 4 & 2 & 1 & b_1 \\ 2 & 5 & 1 & 1 & b_2 \\ 3 & 6 & 0 & 1 & b_3 \end{bmatrix} \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 4 & 2 & 1 & b_1 \\ 0 & -3 & -3 & -1 & b_2 - 2b_1 \\ 0 & -6 & -6 & -2 & b_3 - 3b_1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \begin{bmatrix} 1 & 4 & 2 & 1 & b_1 \\ 0 & -3 & -3 & -1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_1 - 2b_2 + b_3 \end{bmatrix}.$$

The system is consistent iff $b_1 - 2b_2 + b_3 = 0$, or

$$b_1 = 2b_2 - b_3.$$

Therefore,

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = b_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and the set of

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is linearly independent, and $Span\left\{\vec{v}_1, \vec{v}_2\right\} = Span\left\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\right\}.$

Homework 1.7:

#5,11,21,23,27,33,37