1 Section 1.5. Solution Sets of Linear Systems

Consider linear system

$$A\vec{x} = \vec{b}.$$

When $\vec{b} \neq 0$, we say it a non-homogeneous system. We say that (for the same A)

$$A\vec{x} = \vec{0} \quad \text{is the corresponding homogeneous system.}$$

Any homogeneous system is always consistent (why?) $\vec{x} = \vec{0}$ is called the trivial solution. If a homogeneous system admits one non-trivial solution, it must have infinite many non-trivial solutions (why?)

Example 1 *Describe the solution set for the homogeneous system $A\vec{x} = \vec{0}$ where*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

Solution. We perform row reduction to its augmented matrix:

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ -4 & 2 & -6 & 0 \\ -3 & -2 & -7 & 0 \end{bmatrix} \xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + 4R_1 \to R_2} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 14 & 10 & 0 \\ 0 & 7 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2/2 \to R3} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 14 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 5/7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 13/7 & 0 \\ 0 & 1 & 5/7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system corresponding to the last matrix is

$$\begin{aligned} x_1 + (13/7)\, x_3 &= 0 \\ x_2 + (5/7)\, x_3 &= 0, \end{aligned}$$

or $x_1 = -(13/7)\, x_3,\ x_2 = -(5/7)\, x_3$. Therefore,

$$\begin{aligned} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -(13/7)\, x_3 \\ -(5/7)\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -13/7 \\ -5/7 \\ 7/7 \end{bmatrix} \\ &= \frac{x_3}{7} \begin{bmatrix} -13 \\ -5 \\ 7 \end{bmatrix} = t \begin{bmatrix} -13 \\ -5 \\ 7 \end{bmatrix} = t\vec{w}. \end{aligned}$$

This is called parametric representation of solution set. The solution set is $Span\{\vec{w}\}$.

Linearity Principle for Linear System:

Given coefficient matrix A and a vector $\vec{b}$, consider linear system and its homogeneous system

$$A\vec{x} = \vec{0} \tag{1}$$
$$A\vec{x} = \vec{b}. \tag{2}$$

1. If $\vec{x}_h$ and $\vec{y}_h$ both are solutions of homogeneous system (1), then any linear combination of $\vec{x}_h$ and $\vec{y}_h$ is also a solution of the same system (1).

2. If $\vec{x}_h$ is a solution of homogeneous system (1) and if $\vec{y}_{non}$ is a solution of non-homogeneous system (2), then $\vec{x} = \vec{x}_h + \vec{y}_{non}$ is a solution of non-homogeneous system (2).

3. If $\vec{x}_{non}$ and $\vec{y}_{non}$ both are solutions of non-homogeneous system (2), then $\vec{x} = \vec{x}_{non} - \vec{y}_{non}$ is a solution of homogeneous system (1).

Proof. (1) $A(\lambda\vec{x}_h + \delta\vec{y}_h) = \lambda A\vec{x}_h + \delta A\vec{y}_h = \lambda\vec{0} + \delta\vec{0} = \vec{0}$
(2) $A(\vec{x}_h + \vec{y}_{non}) = A\vec{x}_h + A\vec{y}_{non} = \vec{0} + \vec{b} = \vec{b}$
(3) $A(\vec{x}_{non} - \vec{y}_{non}) = A\vec{x}_{non} - A\vec{y}_{non} = \vec{b} - \vec{b} = \vec{0}$. ■

Theorem 2 *Let $\vec{x}_p$ be one particular solution of non-homogeneous system (2). Then any solution of (2) has the form*

$$\vec{x} = \vec{x}_p + \vec{x}_h,$$

where $\vec{x}_h$ is a solution of homogeneous system (1).

> To solve a system $A\vec{x} = \vec{b}$, we follow the following two steps. Step 1: solve the corresponding homogeneous system $A\vec{x} = \vec{0}$, and represent the solution sets of the homogeneous system using parametric vector forms. Step 2: Find a particular solution $\vec{x}_p$ for $A\vec{x} = \vec{b}$. The solution set consists of all sums of $\vec{x}_p$ and a solution from step 1. //

Remark 3 *Finding a particular solution $\vec{x}_p$ of non-homogeneous system (2) could be very challenging. Often it involves sophisticated numerical approximations and large scale computations. We shall only discuss some simple cases. the general discussion is beyond the scope of this course. We shall concentrate on Step 1: solving homogeneous systems.*

1.1 Some examples for representing solution sets of homogeneous systems

Example 4 *Suppose that A has the following reduced Echelon forms. Find the parametric vector form of solution set of $A\vec{x} = \vec{0}$.*

$$(1)\quad A\tilde{}\ \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}. \quad (2)\quad A\tilde{}\ \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3)\quad A\tilde{}\ \begin{bmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. (1) The corresponding equivalent system is

$$\begin{aligned} x_1 + 2x_4 &= 0 \\ x_2 - x_4 &= 0 \\ x_3 + 3x_4 &= 0. \end{aligned}$$

The solution is

$$\begin{aligned} x_1 &= -2x_4 \\ x_2 &= x_4 \\ x_3 &= -3x_{4.} \end{aligned}$$

The parametric form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_{4.} \end{bmatrix} = \begin{bmatrix} -2x_4 \\ x_4 \\ -3x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix} t,$$

where $x_4 = t$ is the parameter, or FREE variable.

Solution. (2) The linear system for

$$A\tilde{} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_3 + 2x_4 &= 0 \\ x_2 + 4x_3 - x_4 &= 0. \end{aligned}$$

Hence

$$\begin{aligned} x_1 &= -3x_3 - 2x_4 \\ x_2 &= -4x_3 + x_4. \end{aligned}$$

Solutions are

$$\begin{aligned} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_{4.} \end{bmatrix} &= \begin{bmatrix} -3x_3 - 2x_4 \\ -4x_3 + x_4 \\ x_3 + 0 \\ 0 + x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} -3 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = t \begin{bmatrix} -3 \\ -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \end{aligned}$$

where $x_3 = t$, $x_4 = s$ are parameters, or FREE variables.

Solution. (3) The system for

$$A\tilde{} \begin{bmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is

$$\begin{aligned} x_1 + 2x_2 - 3x_4 + 2x_5 &= 0 \\ x_3 + 5x_4 - x_5 &= 0, \end{aligned}$$

or

$$\begin{aligned} x_1 &= -2x_2 + 3x_4 - 2x_5 \\ x_3 &= -5x_4 + x_5. \end{aligned}$$

Solution set are

$$\begin{aligned} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} -2x_2 + 3x_4 - 2x_5 \\ x_2 \\ -5x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} \\ &= u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \end{aligned}$$

where $x_2 = u$, $x_4 = v$, $x_5 = w$ are parameters, or free variables.

Remark 5 *From the above examples, we see that each unknown corresponding to non-pivot column represents a parameter or free variable. Therefore, we conclude this section with the following statements*

Conclusion 6 *Consider homogeneous system $A\vec{x} = \vec{0}$, where A be a $m \times n$ matrix, i.e., it has m equations and n unknowns. Then,*

1. *the number of free variables (or parameters) = number of non-pivot columns = number of total columns n - the number of pivots r;*

2. *the homogeneous system has a non-trivial solution $\Longleftrightarrow$ it has infinite many solutions $\Longleftrightarrow$ there is at least one free variable $\Longleftrightarrow$ number of total columns $n > r$;*

3. *in particular, the homogeneous system has non-trivial solutions if* $n > m$ *(i.e., number of columns > number of rows).*

Proof. The first statement is obvious from Remark. The second one is due to the fact that it has non-trivial solutions iff there is at least one free variable. Since the number of pivots r (non-zero leading entries of rows in its Echelon form) cannot exceed the number of rows m (i.e., $m \geq r$), it follows from (1) that if $n > m$, then $n > r$ and the number of free variables $= n - r > 0$. In other words, there is at least one free variable.

■

Homework 1.5:

#1,2,5,,11,13,17,23