Section 1.3. Vector Equations

We now further simplify the process by introducing more general notations. A m-dimensional (column) vector is a $m \times 1$ matrix

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}_{m\times 1}$$

Addition and scalar multiplication are defined in the manner similar to 3-D as follows:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix}$$

$$\lambda\vec{u} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \vdots \\ \lambda u_m \end{bmatrix}.$$

All properties and geometric explanations held for 3-D case extend to m-D without modification.

Definition 1 *Given p vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ in R^m and given p constants $\lambda_1, \lambda_2, ..., \lambda_p$, the vector $\vec{y}$ defined by*

$$\vec{y} = \sum_{i=1}^{p} \lambda_i \vec{u}_i = \lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + ... + \lambda_p\vec{u}_p$$

is called a linear combination of $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ with weights $\lambda_1, \lambda_2, ..., \lambda_p$.

Example 2 *Determine whether $\vec{y}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$, where*

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \ \vec{y} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

Solution 3 *By definition, we need to see if $\vec{y}$ has the form $\vec{y} = \lambda_1\vec{u}_1 + \lambda_2\vec{u}_2$ for some constants λ_1, λ_2. This leads to the equations for λ_1, λ_2 :*

$$\lambda_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

This is the same as the system of linear equations :

$$\begin{aligned} \lambda_1 + 2\lambda_2 &= 7 \\ -2\lambda_1 + 5\lambda_2 &= 4 \\ -5\lambda_1 + 6\lambda_2 &= -3 \end{aligned}$$

whose augmented matrix is

$$\begin{bmatrix} 1 & 2 & \vdots & 7 \\ -2 & 5 & \vdots & 4 \\ -5 & 6 & \vdots & -3 \end{bmatrix}.$$

Its reduced Echelon form is

$$\begin{bmatrix} 1 & 0 & \vdots & 3 \\ 0 & 1 & \vdots & 2 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore, it is consistent, and consequently, $\vec{y}$ *is a linear combination of* $\vec{u}_1$ *and* $\vec{u}_2$. *The weights are the solution* $(\lambda_1, \lambda_2) = (3, 2)$, *i.e.,*

$$\vec{y} = 3\vec{u}_1 + 2\vec{u}_2.$$

Definition 4 *Given p vectors* $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ *in* R^m, *the set of all possible linear combinations of* $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ *is called a subset of* R^m *spanned by* $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$, *and is denoted by*

$$Span\left\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_p\right\}.$$

The above example may be rephrased as follows: determine whether $\vec{y} \in Span\left\{\vec{u}_1, \vec{u}_2\right\}$.

Let us go back to the system and its augmented matrix

$$\left\{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ ...... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{array}\right., \ M = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & | & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & | & b_2 \\ ... & ... & ... & ... & | & ... \\ a_{m1} & a_{m2} & ... & a_{mn} & | & b_m \end{bmatrix}.$$

The $n+1$ column vectors of R^m,

$$\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, ..., \vec{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, ..., \vec{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are called columns of M. Then, the augmented matrix is

$$M = \left[\vec{a}_1, \vec{a}_2, ..., \vec{a}_n, \vec{b}\right],$$

and the coefficient matrix

$$A = \left[\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\right].$$

the system may be expressed as the following vector equation

$$\sum_{i=1}^{n} x_i \vec{a}_i = x_1 \vec{a}_1 + x_2 \vec{a}_2 + ... + x_n \vec{a}_n = \vec{b}.$$

Now, system of linear equations, augmented matrix, vector equation all mean the same thing.

$$\sum_{i=1}^{n} x_i \vec{a}_i = x_1 \vec{a}_1 + x_2 \vec{a}_2 + ... + x_n \vec{a}_n = \vec{b} \text{ is consistent}$$

iff (if and only if)

$$\vec{b} \text{ is a linear combination of } \{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$$

iff

$$\vec{b} \text{ belongs to the subspace spanned by } \vec{a}_1, \vec{a}_2, ..., \vec{a}_n$$
$$\text{or} \quad \vec{b} \in Span \{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}.$$

Homework:

#5, 9, 11, 13, 15, 19, 21, 25