Section 1.2. Row reductions, Echelon forms

1.2.1. Echelon Forms

The first entry of a row (or column) is called **the leading entry** of the row (or column).

Definition 1 *A matrix is called in **Echelon form** (upper triangle form) if*

(a) All non-zero rows are above any zero-row (row with all entries zero),

(b) For any two rows, the column containing the leading entry of the upper row is on the left of the column containing the leading entry of the lower row.

$$\text{Echelon form: } \begin{bmatrix} 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & -8 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 3 & 0 \\ 0 & 4 & 1 & 1 & 8 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}$$

$$\text{non-Echelon form: } \begin{bmatrix} 0 & -2 & 1 & 3 & 0 \\ 2 & 0 & -8 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 2 *If, in addition to (a) and (b) above,*

*(c) all leading entries are 1 and it is the only non-zero entry in the column, the matrix is called **Reduced Echelon form**.*

$$\text{Reduced Echelon form: } \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem 3 *Any matrix can be reduced by elementary row operations to the unique reduced Echelon form. The solution set of a system is the same as that of the system from the reduced Echelon form of the augmented matrix.*

Definition 4 *All leading entries 1 are called **pivot positions**. All columns containing pivot positions are called **pivot columns**.*

The example in the end of last section can be summed up as

1.2.2. Gauss-Jordan Algorithm

Gauss-Jordan Algorithm of finding reduced Echelon form:

Step 1. From the left, find the first non-zero column (it is a pivot column). By interchanging two rows if necessary, make sure that the leading entry of the column is non-zero. We end up with the following matrix (* represents any number)

$$\begin{bmatrix} 0 & a & * & * & ... \\ 0 & b & * & * & ... \\ ... & ... & ... & ... & ... \\ 0 & c & * & * & ... \end{bmatrix}, \ a \neq 0.$$

Step 2. Perform elementary row operation #3: $R_1/a \to R_1$, we arrive at

$$\begin{bmatrix} 0 & 1 & * & * & ... \\ 0 & b & * & * & ... \\ ... & ... & ... & ... & ... \\ 0 & c & * & * & ... \end{bmatrix}$$

Then, perform elementary row operation #1 several times (i.e., first, $R_2 - bR_1 \to R_2$, ..., finally $R_m - cR_1 \to R_m$), we obtain a matrix

$$\begin{bmatrix} 0 & 1 & * & * & ... \\ 0 & 0 & A & B & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & C & * & ... \end{bmatrix}$$

Step 3. repeat the above two steps for the submatrix obtained from deleting the first row:

$$\begin{bmatrix} 0 & 0 & A & B & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & C & * & ... \end{bmatrix}$$

The matrix is then reduced row-equivalently to

$$\begin{bmatrix} 0 & 1 & W & * & ... \\ 0 & 0 & 1 & * & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & * & ... \end{bmatrix}$$

By perform row operation: $R_1 - WR_2 \to R_1$, we have

$$\begin{bmatrix} 0 & 1 & 0 & * & ... \\ 0 & 0 & 1 & * & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & * & ... \end{bmatrix}.$$

Step 4. Repeat till reduced Echelon form.

Example 5 *Find the reduced Echelon form for*

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ -4 & 5 & 9 & | & -9 \end{bmatrix}$$

*and then solve (**??**):*

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

Solution: We perform row operations as follows:

$$\begin{bmatrix} 1 & -2 & 1 & \vert & 0 \\ 0 & 2 & -8 & \vert & 8 \\ -4 & 5 & 9 & \vert & -9 \end{bmatrix} \xrightarrow{R_3+4R_1\to R_3} \begin{bmatrix} 1 & -2 & 1 & \vert & 0 \\ 0 & 2 & -8 & \vert & 8 \\ 0 & -3 & 13 & \vert & -9 \end{bmatrix}$$

$$\xrightarrow{R_2/2\to R_2} \begin{bmatrix} 1 & -2 & 1 & \vert & 0 \\ 0 & 1 & -4 & \vert & 4 \\ 0 & -3 & 13 & \vert & -9 \end{bmatrix} \xrightarrow{R_1+2R_2\to R_1} \begin{bmatrix} 1 & 0 & -7 & \vert & 8 \\ 0 & 1 & -4 & \vert & 4 \\ 0 & -3 & 13 & \vert & -9 \end{bmatrix}$$

$$\xrightarrow{R_3+3R_2\to R_3} \begin{bmatrix} 1 & 0 & -7 & \vert & 8 \\ 0 & 1 & -4 & \vert & 4 \\ 0 & 0 & 1 & \vert & 3 \end{bmatrix} \xrightarrow{R_1+7R_3\to R_1} \begin{bmatrix} 1 & 0 & 0 & \vert & 29 \\ 0 & 1 & -4 & \vert & 4 \\ 0 & 0 & 1 & \vert & 3 \end{bmatrix}$$

$$\xrightarrow{R_2+4R_3\to R_2} \begin{bmatrix} 1 & 0 & 0 & \vert & 29 \\ 0 & 1 & 0 & \vert & 16 \\ 0 & 0 & 1 & \vert & 3 \end{bmatrix}$$

Solution $x_1 = 29$, $x_2 = 16$, $x_3 = 3$.

Theorem 6 *(Existence & Uniqueness) A system is called consistent if it admits at least one solution. A system is consistent if and only if the rightmost column contains NO pivot. in other words, there is no row having the form [0,0,...,0,b]* $(b \neq 0)$.

Example 7 *In the second example of the previous section, we have*

$$\begin{bmatrix} 0 & 1 & -4 & \vert & 8 \\ 2 & -3 & 2 & \vert & 1 \\ 5 & -8 & 7 & \vert & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & 1 & \vert & 1/2 \\ 0 & 1 & -4 & \vert & 8 \\ 0 & 0 & 0 & \vert & 5/2 \end{bmatrix}.$$

The last column is a pivot column. Thus, it is inconsistent.

Example 8 *Determine if the system*

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Sol. The augmented matrix

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & \vert & -5 \\ 3 & -7 & 8 & -5 & 8 & \vert & 9 \\ 3 & -9 & 12 & -9 & 6 & \vert & 15 \end{bmatrix}$$

Row operations:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & \vert & -5 \\ 3 & -7 & 8 & -5 & 8 & \vert & 9 \\ 3 & -9 & 12 & -9 & 6 & \vert & 15 \end{bmatrix} \xrightarrow{R_2\to R_1} \begin{bmatrix} 3 & -7 & 8 & -5 & 8 & \vert & 9 \\ 0 & 3 & -6 & 6 & 4 & \vert & -5 \\ 3 & -9 & 12 & -9 & 6 & \vert & 15 \end{bmatrix}$$

$$\xrightarrow{R_3-R_1\to R_3} \begin{bmatrix} 3 & -7 & 8 & -5 & 8 & \vert & 9 \\ 0 & 3 & -6 & 6 & 4 & \vert & -5 \\ 0 & -2 & 4 & -4 & -2 & \vert & 6 \end{bmatrix} \xrightarrow{R_3+\frac{2}{3}R_2\to R_3} \begin{bmatrix} 3 & -7 & 8 & -5 & 8 & \vert & 9 \\ 0 & 3 & -6 & 6 & 4 & \vert & -5 \\ 0 & 0 & 0 & 0 & 2/3 & \vert & * \end{bmatrix}$$

Answer: consistent.

Homework:

#3, 11, 13, 15, 19, 21