MTH 253: Elementary Linear Algebra

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Sec. 1.1. Systems of Linear Equations

1.1.1. 2D examples

Example 1 *Consider a system of two equations with two unknowns:*

$$\begin{cases} x - 2y = -1 \\ -x + 3y = 3 \end{cases}. \tag{1}$$

There are two commonly used methods to solve linear systems–elimination method and substitution method. We now use the elimination method by adding both equation to obtain

$$y = 2.$$

Substitute into the first equation of system (1), we find

$$x = 2y - 1 = 3.$$

A solution of system (1) is an ordered couple (x_0, y_0) *that satisfies both equations. Here, the system has exact one solution* $(3, 2)$.

Example 2 *Consider another system*

$$\begin{cases} x - 2y = -1 \\ -x + 2y = 3 \end{cases}. \tag{2}$$

Adding both equations leads to

$$0 = 2.$$

This contradictory equation indicates that system (2) cannot possibly have a solution.

Example 3 *Consider another system*

$$\begin{cases} x - 2y = -1 \\ -x + 2y = 1 \end{cases}. \tag{3}$$

The second equation is a multiple of the first equation (by -1). Therefore, this system has only one independent equation. Consequently, there are infinitely many solutions:

$$x = -1 + 2t$$
$$y = t$$

for any choice of t.

Geometrically, in 2D, the graph of a linear equation is a straight line. A solution of a system of two equation is an intersection of two lines. There are tow possible situations for two lines: (a) two lines intersect each other (one solution), (b) two lines are parallel (no solution), and (c) two line are identical (any point on the line is a solution).

This observation and the elimination method extends to general situations.

1.1.2. General systems.

A system of m equations with n variables, denoted by $x_1, x_2, ..., x_n$, reads as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ ...... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \tag{4}$$

a_{ij} is called a coefficient. It is in the ith equation and is associated with x_j.

A solution of system (4) consists of n ordered numbers $(x_1, x_2, ..., x_n)$ satisfying all m equations. The set of all possible solutions is called a solution set.

Example 4 *A system of* 3 *equations with* 3 *unknowns:*

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 3x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \tag{5}$$

We use the same method of elimination: Equation #3 is replaced by the sum of itself and 4 times Equation #1 (while maintain the other equations):

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 3x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases} . \tag{6}$$

Next, We add Equation #2 to #3 to arrive at

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 3x_2 - 8x_3 = 8 \\ 5x_3 = -1 \end{cases} .$$

We solve x_3 *from the last equation:* $x_3 = -\frac{1}{5}$. *Substitute it into Equation #2, we find*

$$3x_2 - 8\left(-\frac{1}{5}\right) = 8 \Longrightarrow 3x_2 = 8 - \frac{8}{5} = \frac{32}{5}.$$

Finally, substitute $x_3 = -\frac{1}{5}$, $x_2 = \frac{32}{15}$ *into the first equation:*

$$x_1 - 2\left(\frac{32}{15}\right) + \left(-\frac{1}{5}\right) = 0 \Longrightarrow x_1 = \frac{67}{15}.$$

This whole process can be simplified using symbolic means. We introduce the coefficient matrix for system (4)

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}_{m\times n} . \tag{7}$$

It has m rows and n columns. it is also called a $m \times n$ matrix. The entire information of system (4) can be found in

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & | & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & | & b_2 \\ ... & ... & ... & ... & | & ... \\ a_{m1} & a_{m2} & ... & a_{mn} & | & b_m \end{bmatrix}. \tag{8}$$

It is called Augmented matrix. This is a $m \times (n+1)$ matrix.

The elimination method basically consists of the following row operations:

Definition 5 *The following operations are called elementary row operations:*

1. *Replace one row by the sum of itself and a multiple of another row;*
2. *Interchange two rows;*
3. *One row is replaced by a non-zero multiple of itself.*

Theorem 6 *Elementary row reductions do not alter the solution set of any system of linear equations.*

Example 7 *Solve system (5) using row operations.*

Solution: Augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 3 & -8 & | & 8 \\ -4 & 5 & 9 & | & -9 \end{bmatrix}$$

We now perform a series of row operation in the way equivalent to what we did before:

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 3 & -8 & | & 8 \\ -4 & 5 & 9 & | & -9 \end{bmatrix} \begin{matrix} \times 4 \\ \\ \\ \end{matrix} \xrightarrow{4\times R_1 + R_3 \to R_3} \begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 3 & -8 & | & 8 \\ 0 & -3 & 13 & | & -9 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 3 & -8 & | & 8 \\ 0 & 0 & 5 & | & -1 \end{bmatrix}$$

The corresponding system is

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad 3x_2 - 8x_3 = 8 \\ \qquad 5x_3 = -1 \end{cases}$$

which is exactly the same as before. We can continue row operations:

$$\left[\begin{array}{ccccc} 1 & -2 & 1 & | & 0 \\ 0 & 3 & -8 & | & 8 \\ 0 & 0 & 5 & | & -1 \end{array}\right] \xrightarrow{R_3/5 \to R_3} \left[\begin{array}{ccccc} 1 & -2 & 1 & | & 0 \\ 0 & 3 & -8 & | & 8 \\ 0 & 0 & 1 & | & -\frac{1}{5} \end{array}\right].$$

$$\xrightarrow{8 \times R_3 + R_2 \to R_2} \left[\begin{array}{ccccc} 1 & -2 & 1 & | & 0 \\ 0 & 3 & 0 & | & \frac{32}{5} \\ 0 & 0 & 1 & | & -\frac{1}{5} \end{array}\right] \xrightarrow{R_2/3 \to R_2} \left[\begin{array}{ccccc} 1 & -2 & 1 & | & 0 \\ 0 & 1 & 0 & | & \frac{32}{15} \\ 0 & 0 & 1 & | & -\frac{1}{5} \end{array}\right] \times 2$$

$$\xrightarrow{2 \times R_2 + R_1 \to R_1} \left[\begin{array}{ccccc} 1 & 0 & 1 & | & \frac{64}{15} \\ 0 & 1 & 0 & | & \frac{32}{15} \\ 0 & 0 & 1 & | & -\frac{1}{5} \end{array}\right] \xrightarrow{(-1) \times R_3 + R_1 \to R_1} \left[\begin{array}{ccccc} 1 & 0 & 0 & | & \frac{64}{15} + \frac{1}{5} \\ 0 & 1 & 0 & | & \frac{32}{15} \\ 0 & 0 & 1 & | & -\frac{1}{5} \end{array}\right]$$

$$= \left[\begin{array}{ccccc} 1 & 0 & 0 & | & \frac{67}{15} \\ 0 & 1 & 0 & | & \frac{32}{15} \\ 0 & 0 & 1 & | & -\frac{1}{5} \end{array}\right] \text{(Reduced Echelon Form)} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{67}{15} \\ \frac{32}{15} \\ -\frac{1}{5} \end{bmatrix}.$$

Example 8 *Solve*

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \tag{9}$$

Solution: Write down the Matrix form and perform row operations:

$$\left[\begin{array}{ccccc} 0 & 1 & -4 & | & 8 \\ 2 & -3 & 2 & | & 1 \\ 5 & -8 & 7 & | & 1 \end{array}\right] \xrightarrow{R_2 \to R_1} \left[\begin{array}{ccccc} 2 & -3 & 2 & | & 1 \\ 0 & 1 & -4 & | & 8 \\ 5 & -8 & 7 & | & 1 \end{array}\right]$$

$$\xrightarrow{R_1/2 \to R_1} \left[\begin{array}{ccccc} 1 & -3/2 & 1 & | & 1/2 \\ 0 & 1 & -4 & | & 8 \\ 5 & -8 & 7 & | & 1 \end{array}\right] -5 \xrightarrow{(-5)\, R_1 + R_3 \to R_3} \left[\begin{array}{ccccc} 1 & -3/2 & 1 & | & 1/2 \\ 0 & 1 & -4 & | & 8 \\ 0 & -1/2 & 2 & | & -3/2 \end{array}\right] 1/2$$

$$\xrightarrow{(-1/2)\, R_2 + R_3 \to R_3} \left[\begin{array}{ccccc} 1 & -3/2 & 1 & | & 1/2 \\ 0 & 1 & -4 & | & 8 \\ 0 & 0 & 0 & | & 5/2 \end{array}\right].$$

We convert back to the system:

$$\begin{cases} x_1 - \dfrac{3}{2}x_2 + x_3 = \dfrac{1}{2} \\ x_2 - 4x_3 = 8 \\ 0 = \dfrac{5}{2} \end{cases} \implies \textit{impossible, means no solution.}$$

Homework:

#7, 11, 13, 15, 17, 21, 23, 25