Section 3.1 Calculus of Vector-Functions

**Definition.** A vector-valued function is a rule that assigns a vector to each member in a subset of $\mathbb{R}^3$. In other words, a vector-valued function is an ordered triple of functions, say $f(t), g(t), h(t)$, and can be expressed as

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle.$$

For instance,

$$\vec{r}(t) = \langle 1 + t, 2t, 2 - t \rangle$$
$$\vec{q}(t) = \left\langle \frac{1}{t - 1}, \ln(t), \sqrt{2 - t} \right\rangle$$

are vector-valued functions. The domain of a vector-valued function is a subset of all real number at which the function is well-defined, i.e.,

**Domain of $\vec{r}(t)$**

$$\begin{align*}
\text{Domain of } \vec{r}(t) &= \{ t \mid \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \text{ is defined} \} \\
&= \{ t \mid \text{each of } f(t), g(t), h(t) \text{ is defined} \} \\
&= \{ t \mid f(t) \text{ is defined} \} \cap \{ t \mid g(t) \text{ is defined} \} \cap \{ t \mid h(t) \text{ is defined} \}.
\end{align*}$$

So

$$D(\vec{r}) = D(f) \cap D(g) \cap D(h).$$

Any vector-valued function $\vec{r}(t) = \langle x, y, z \rangle$ may be written in terms of its components as

$$\begin{align*}
x &= f(t) \\
y &= g(t) \\
z &= h(t).
\end{align*}$$

Thus, the graph of a vector-valued function is a parametric curve in space. For instance, the function

$$\vec{r}(t) = \langle 1 + t, 2t, 2 - t \rangle$$

is defined for all $t$. Its component form is

$$\begin{align*}
x &= 1 + t \\
y &= 2t \\
z &= 2 - t.
\end{align*}$$
The graph is a straight line with a direction \((1, 2, -1)\) passing through \((1, 0, 2)\).

**Example 1.1.** Find the domain of
\[
\vec{r}(t) = \left\langle \frac{1}{t - 1}, \ln(t), \sqrt{2 - t} \right\rangle.
\]

Sol: We know that
\[
D\left(\frac{1}{t - 1}\right) = \{t \neq 1\} = (-\infty, 1) \cup (1, \infty)
\]
\[
D(\ln(t)) = \{t > 0\} = (0, \infty)
\]
\[
D(\sqrt{2 - t}) = \{t \leq 2\} = (-\infty, 2].
\]

So
\[
D(\vec{r}) = D\left(\frac{1}{t - 1}\right) \cap D(\ln(t)) \cap D(\sqrt{2 - t})
\]
\[
= ((-\infty, 1) \cup (1, \infty)) \cap (0, \infty) \cap (-\infty, 2]
\]
\[
= ((-\infty, 1) \cup (1, \infty)) \cap (0, 2]
\]
\[
= (0, 1) \cup (1, 2].
\]

Limits of vector-valued functions are defined through components:

For any vector-valued function \(\vec{r}(t) = (f(t), g(t), h(t))\), the limit
\[
\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle
\]
exists if and only if the limits of all three components exist.

**Example 1.2.** Consider
\[
\vec{r}(t) = (2 \cos t, \sin t, t).
\]
(a) Find
\[
\lim_{t \to 0} \vec{r}(t), \quad \lim_{t \to \pi/2} \vec{r}(t).
\]

(b) Discuss and sketch its graph.

Solution: (a)
\[
\lim_{t \to 0} \vec{r}'(t) = \left\langle \lim_{t \to 0} (2 \cos t), \lim_{t \to 0} \sin t, \lim_{t \to 0} t \right\rangle = \langle 2, 0, 0 \rangle
\]
\[
\lim_{t \to \pi/2} \vec{r}'(t) = \left\langle \lim_{t \to \pi/2} (2 \cos t), \lim_{t \to \pi/2} \sin t, \lim_{t \to \pi/2} t \right\rangle = \langle 0, 1, \pi \rangle.
\]

(b) Let us first take a look at the projection of the curve onto \(xy\)-plane
\[
x = 2 \cos t
\]
\[
y = \sin t.
\]

We know that its graph is an ellipse

\[
t = \frac{\pi}{4} \text{ is the angle to } x-axis
\]

In 3D, as \(t\) increases from \(t = 0\), the curve starting at \((2, 0, 0)\) on \(xy\)-plane, moves in the way that its first two component \((x, y)\) moving along the ellipse in the above figure counter-clockwise while its \(z\)- component increases linearly, as if we raise vertically the ellipse. The curve is on the elliptic cylinder, and is called elliptic helix.
**Definition.** For any vector-valued function \( \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \), if the limit of the difference quotation

\[
\lim_{h \to 0} \frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}
\]

exists, we say \( \vec{r}(t) \) is differentiable at \( t = t_0 \). In this case, we call the limit the derivative at \( t = t_0 \) and denote it by \( \vec{r}'(t_0) \) or

\[
\frac{d\vec{r}}{dt}(t_0) = \vec{r}'(t_0) = \lim_{h \to 0} \frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}.
\]

We can show that \( \vec{r}(t) \) is differentiable at \( t = t_0 \) if and only if all three components are differentiable and

\[
\vec{r}'(t_0) = \langle f'(t_0), g'(t_0), h'(t_0) \rangle.
\]

The derivative vector for any \( t \), \( \vec{r}'(t) \), is again a vector-valued function. Higher order derivatives are then defined accordingly. For instance,

\[
\vec{r}''(t) = \langle f''(t), g''(t), h''(t) \rangle.
\]

Geometrically,

\[
\vec{r}(t_0 + h) - \vec{r}(t_0)
\]

represents the vector from \( \vec{r}(t_0) \) to \( \vec{r}(t_0 + h) \). So for any small \( h > 0 \),

\[
\frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}
\]
is a normalized (otherwise, the length of \( \vec{r}(t_0 + h) - \vec{r}(t_0) \) would be a very small) secant direction. Therefore, the limit vector is "tangent" to the curve at \( t = t_0 \).

**Definition.** We call

\[
\vec{r}'(t_0) = (f'(t_0), g'(t_0), h'(t_0))
\]

the tangent vector of the parametric curve \( \vec{r}(t) \) at \( t = t_0 \), and

\[
\vec{T}(t_0) = \frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}
\]

the unit tangent vector. A curve \( \vec{r}(t) \) is called smooth if \( \vec{r}'(t) \) exists and \( \vec{r}'(t) \neq 0 \).

**Example 1.3.** Consider a circular helix

\[
\vec{r}(t) = (\cos t, \sin t, t).
\]

Find \( \vec{r}'(t) \), \( \vec{T}(t) \), and \( \vec{r}''(t) \). Find also \( \vec{r}'(0) \), \( \vec{T}(0) \).
Solution:
\[
\begin{align*}
\vec{r}'(t) &= \langle -\sin t, \cos t, 1 \rangle \\
\vec{r}''(t) &= \langle -\cos t, -\sin t, 0 \rangle \\
\vec{T}(t) &= \frac{1}{|\vec{r}'(t)|} \vec{r}''(t) \\
&= \frac{1}{\sqrt{\sin^2 t + \cos^2 t + 1}} \langle -\sin t, \cos t, 1 \rangle = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle \\
\vec{r}'(0) &= \langle 0, 1, 1 \rangle \\
\vec{T}(0) &= \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle.
\end{align*}
\]

**Properties of derivatives:** \( \lambda \) is a scalar constant, \( f(t) \) is a scalar function.

1. Addition:
\[
\left(\vec{u} (t) + \vec{v}(t)\right)' = \vec{u}'(t) + \vec{v}'(t)
\]

2. Scalar function multiplication:
\[
(f(t) \vec{u}(t))' = f'(t) \vec{u}'(t) + f(t) \vec{u}'(t)
\]

3. Scalar (constant) multiplication:
\[
(\lambda \vec{u}(t))' = \lambda \vec{u}'(t)
\]

4. Dot product:
\[
(\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)
\]

5. Cross product:
\[
(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)
\]

6. Chain rule:
\[
\frac{d}{dt} \vec{u}(f(t)) = \left( \frac{d\vec{u}}{dt}(f(t)) \right) \frac{df}{dt}(t) = \vec{u}'(f(t)) f'(t)
\]
All above properties can be verified by direction computations.

As in the case of one variable functions, derivative $\vec{r}'(t_0)$ measures the rate (vector) at which function $\vec{r}(t)$ changes across $t = t_0$. Thus

$$ |\vec{r}'(t_0)| \text{ is the magnitude of the rate of change }$$

$$ \vec{T}(t_0) \text{ is the direction of change }$$

Note that since

$$ |\vec{r}(t)| = \sqrt{\vec{r}(t) \cdot \vec{r}(t)} = (\vec{r}(t) \cdot \vec{r}(t))^{\frac{1}{2}} $$

we have

$$ \frac{d}{dt} |\vec{r}(t)| = \frac{1}{2} (\vec{r}(t) \cdot \vec{r}(t))^{-\frac{1}{2}} (\vec{r}(t) \cdot \vec{r}(t))' $$

$$ = \frac{1}{2} |\vec{r}(t)|^{-1} (\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t)) $$

$$ = \frac{\vec{r}'(t) \cdot \vec{r}(t)}{|\vec{r}(t)|}. $$

This shows that in general,

$$ \frac{d}{dt} |\vec{r}(t)| \neq |\vec{r}'(t)|, $$

i.e.,

Rate of change for $|\vec{r}(t)| \neq$ Magnitude of rate of change for $\vec{r}(t)$.

In physics, if $\vec{r}(t)$ describes the position of a moving object, then

$$ \vec{v}(t) = \vec{r}'(t) \quad \text{is velocity} $$

$$ v(t) = |\vec{v}(t)| \quad \text{is speed} $$

$$ \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) \quad \text{is acceleration.} $$

**Definition.** Integrals, indefinite and definite, are defined accordingly:

$$ \int \vec{r}(t) \, dt = \left\langle \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right\rangle $$

$$ \int_a^b \vec{r}(t) \, dt = \left\langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \right\rangle. $$
Note that for indefinite integrals, we always end up a constant vector \( \vec{C} = \langle C_1, C_2, C_3 \rangle \):

\[
\int \vec{r}(t) \, dt = \left\langle \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right\rangle + \vec{C}.
\]

**Example 1.4.** Consider

\[
\vec{r}(t) = 1 + t^3, te^{-t}, \sin (2t) \rangle.
\]

Find (a) \( \vec{r}'(t) \), and (b) equations of the tangent at \( t = 0 \).

Solution: (a)

\[
\vec{r}'(t) = 3t^2, e^{-t} - te^{-t}, 2 \cos (2t) \rangle.
\]

(b) The tangent line passes through the terminal point of the vector \( \vec{r}'(0) = \langle 1, 0, 0 \rangle \), i.e., passing through \( (1, 0, 0) \) with direction

\[
\vec{r}'(0) = \langle 0, 1, 2 \rangle.
\]

So the equations are

\[
\begin{align*}
  x &= 1 \\
  y &= t \\
  z &= 2t.
\end{align*}
\]

**Example 1.5.** Find (a) \( \int \vec{r}(t) \, dt \) and (b) \( \int_0^\pi \vec{r}(t) \, dt \) if

\[
\vec{r}(t) = 2 \cos t, \sin t, 3t^2 \rangle.
\]

Solution: (a)

\[
\int \vec{r}(t) \, dt = \left\langle \int 2 \cos t \, dt, \int \sin t \, dt, \int 3t^2 \, dt \right\rangle
\]  

\[
= 2 \sin t + C_1, - \cos t + C_2, t^3 + C_3 \rangle
\]

\[
= 2 \sin t, - \cos t, t^3 \rangle + \vec{C}
\]

where

\[
\vec{C} = \langle C_1, C_2, C_3 \rangle \text{ is an arbitrary constant vector.}
\]
(b) According to Fundamental Theorem of Calculus,
\[ \int_0^\pi \vec{r}(t) \, dt = \left. 2 \sin t, -\cos t, t^3 \right|_{t=0}^{t=\pi} \]
\[ = 2 \sin \pi, -\cos \pi, \pi^3 \right) - \langle 2 \sin 0, -\cos 0, 0 \rangle \]
\[ = 0, 2, \pi^3 \right) \]

Homework:
1. Find domain and limit.
   (a) \( \vec{r}(t) = \left\langle \frac{t - 1}{t + 1}, \sqrt{t}, \sin (\pi (t^2 + 1)) \right\rangle \), \( \lim_{t \to 1} \vec{r}(t) =? \)
   (b) \( \vec{r}(t) = \left\langle \arctan t, e^{-t^2}, \frac{\ln t}{t} \right\rangle \), \( \lim_{t \to \infty} \vec{r}(t) =? \)

2. Sketch the curve. Indicate with an arrow the direction in which \( t \) increases.
   (a) \( \vec{r}(t) = \langle \cos 2t, t, \sin 2t \rangle \)
   (b) \( \vec{r}(t) = \langle 1 + 2t, t, -3t \rangle \)

3. Find a vector equation for the curve of intersection of two surfaces.
   (a) \( x^2 + y^2 = 4 \) and \( z = xy \)
   (b) \( z = 2x^2 + y^2 \) and \( y = x^2 \)

4. Find the derivative.
   (a) \( \vec{r}(t) = \left\langle t^2, \cos 3t, e^{-t^2} \right\rangle \)
   (b) \( \vec{r}(t) = t^2 \vec{a} \times \left( e^t \vec{b} + 2t \vec{c} \right), \ \vec{a}, \vec{b}, \) and \( \vec{c} \) are three constant vectors.

5. Find the integral.
   (a) \( \int (\sin \pi t, \cos \pi t, e^{2t}) \, dt \)
   (b) \( \int_0^1 \left( 8t^2 \vec{i} + 9t^2 \vec{j} + 25t^4 \vec{k} \right) \, dt \)
6. Find (i) unit tangent $\mathbf{T}$ at given point and (ii) equation of tangent line to the curve at that point.

(a) $\mathbf{r}'(t) = \langle 2e^{-t} \cos t, \ e^{-t} \sin t, \ e^{-t} \rangle$; \ (2,0,1)
(b) $\mathbf{r}'(t) = \ln t, \ 2\sqrt{t}, \ t^2 \rangle$; \ (0,2,1)

7. The angle between two curves at a point of intersection is defined as the angle between their tangents. Find the point of intersection and the angle between $\mathbf{r}_1 = \langle t, \ 1-t, \ 3+t^2 \rangle$ and $\mathbf{r}_2 = \langle 3-s, \ s-2, \ s^2 \rangle$. 

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