Section 2.4: Equations of Lines and Planes

An equation of three variable $F(x, y, z) = 0$ is called an equation of a surface S if

$$(x_1, y_1, z_1) \in S \text{ if and only if } F(x_1, y_1, z_1) = 0.$$

For instance,

$$x^2 + y^2 + z^2 = 1$$

is the equation of the unit sphere centered at the origin. The graph of a system of two equations

$$F(x, y, z) = 0, \; G(x, y, z) = 0$$

represents the intersection of two surfaces represented by $F(x, y, z) = 0$ and by $G(x, y, z) = 0$, respectively, and is usually a curve.

A) Lines in R^3.

A line l is determined by two elements: one point P_0 on the line l and a direction $\vec{v}$ of l,i.e., any vector that is parallel to l. The goal here is to describe the line using algebra so that one is able to digitize it. Suppose that the coordinate of the point P_0 on the line and a direction $\vec{v}$ are given as:

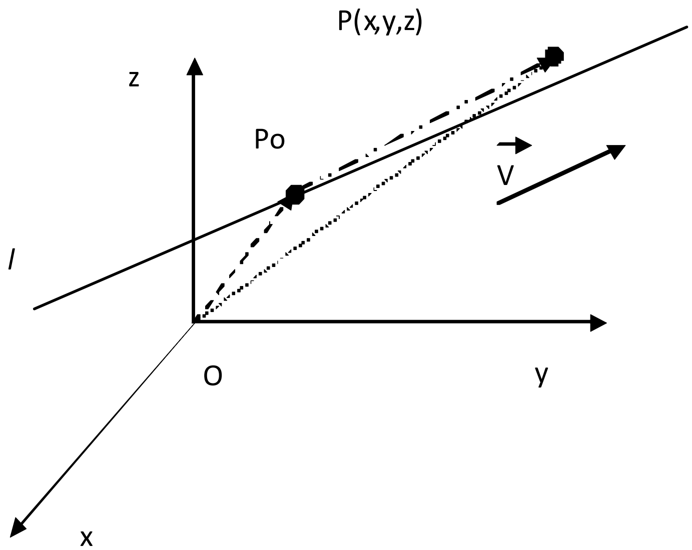
$$\begin{aligned} P_0(x_0, y_0, z_0) & \text{ is a given point on } l \\ \vec{v} = \langle a, b, c \rangle & \text{ is parallel to } l. \end{aligned}$$

Consider any point $P(x, y, z)$ in the space. Let $\overrightarrow{P_0P}$ be the vector connecting P_0 and P. If P is located exactly on the line, then $\overrightarrow{P_0P}$ is parallel to the line l,and thus it is parallel to $\vec{v}$. On the other hand, if P is off the line, then, since P_0 is on the line, $\overrightarrow{P_0P}$ cannot possibly be parallel to the line. Therefore, $\overrightarrow{P_0P}$ cannot possibly be parallel to $\vec{v}$. We just concluded that

$$P \text{ is on } l \text{ if and only if } \overrightarrow{P_0P} \text{ is parallel to } \vec{v}.$$

Now

$$\begin{aligned} \overrightarrow{P_0P} &= \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = \langle x - x_0, y - y_0, z - z_0 \rangle \\ \vec{v} &= \langle a, b, c \rangle . \end{aligned}$$



So

$$\overrightarrow{P_0P} \ \ // \ \vec{v} \quad \Longleftrightarrow \quad \overrightarrow{P_0P} = t\vec{v} \ \ \text{(for a constant } t\text{)}$$

which is equivalent to

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We called these three equation *symmetric form* of the system of equations for line l.

If we set

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t,$$

which is equivalent to

$$\frac{x - x_0}{a} = t$$
$$\frac{y - y_0}{b} = t$$
$$\frac{z - z_0}{c} = t,$$

Or

$$\begin{aligned}x &= x_0 + at\\ y &= y_0 + bt\\ z &= z_0 + ct,\end{aligned}$$

We call it the *parametric form* of the system of equations for line l. This system can be written in the form of vector equation:

$$\vec{r} = \overrightarrow{r_0} + t\vec{v}, \qquad \vec{r} = \langle x, y, z\rangle\,,\ \overrightarrow{r_0} = \langle x_0, y_0, z_0\rangle\,.$$

Example 4.1. (a) Find the equation of the line passing through $(5, -1, 3)$, having direction $\vec{v} = \langle 1, 0, -2\rangle$. Express answer in (i) symmetric form, (ii) vector form, and (iii) parametric form.

(b) Find two other points on the line.

Solution: (a) (i)

$$\frac{x-5}{1} = \frac{y+1}{0} = \frac{z-3}{-2}$$

(ii)

$$\vec{r} = \langle 5, -1, 3\rangle + t\,\langle 1, 0, -2\rangle$$

(iii)

$$\begin{aligned}x &= 5 + t\\ y &= -1\\ z &= 3 - 2t\end{aligned}$$

(b) Take $t = 1$, $(x, y, z) = (6, -1, 1)$. Take $t = -1$, $(x, y, z) = (4, -1, 5)$.

Example 4.2. (a) Find the equation, in symmetric form, of the line l passing through $A\,(2, 4, -3)$ and $B\,(3, -1, 1)$. (b) Determine where the line l intersects $xy - plane$.

Solution. (a) The line is parallel to vector $\overrightarrow{AB}$. So we choose this vector as the direction of l,

$$\vec{v} = \overrightarrow{AB} = \langle 3, -1, 1\rangle - \langle 2, 4, -3\rangle = \langle 1, -5, 4\rangle\,,$$

and $A\,(2, 4, -3)$ as the point on l. Thus, the equation is

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}.$$

(b) If this line crosses $xy - plane$ somewhere at (x, y, z), then $z = 0$. So this point $(x, y, 0)$ satisfies the line equation, i.e.,

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{0+3}{4}.$$

We solve this system to obtain

$$\begin{aligned} x &= 2 + \frac{3}{4} = \frac{11}{4} \\ y &= 4 - 5\left(\frac{3}{4}\right) = \frac{1}{4} \\ z &= 0. \end{aligned}$$

Example 4.3. Given two lines

$$\begin{aligned} l_1 &: \ x = 1 + t, \ y = -2 + 3t, \ z = 4 - t \\ l_2 &: \ x = 2t, \ y = 3 + t, \ z = -3 + 4t. \end{aligned}$$

Determine whether they intersect each other, or they are parallel, or neither (skew lines).

Solution: First of all, in each line equation, "t" is a parameter (or free variable) that can be chosen arbitrarily. Therefore, the parameter "t" in the equations for line l_1 is DIFFERENT from the parameter "t" in the equations for line l_2. To clarify this issue, we rewrite as

$$\begin{aligned} l_1 &: \ x = 1 + t, \ y = -2 + 3t, \ z = 4 - t \\ l_2 &: \ x = 2s, \ y = 3 + s, \ z = -3 + 4s, \end{aligned}$$

and intersection of these two lines consists of solutions of the following system of six equations,

$$\begin{aligned} x = 1 + t, \ y = -2 + 3t, \ z = 4 - t \\ x = 2s, \ y = 3 + s, \ z = -3 + 4s, \end{aligned}$$

for five variables: x, y, z, t, s. Two lines intersect each other if and only If this system has a solution. If, for instance, $(x_0, y_0, z_0, t_0, s_0)$ is a solution, then the first three components, (x_0, y_0, z_0) is a point of intersection.

We now proceed to solution the system by eliminating x,y,z:

$$\begin{aligned} 1 + t &= 2s && (1) \\ -2 + 3t &= 3 + s && (2) \\ 4 - t &= -3 + 4s. && (3) \end{aligned}$$

There are three equations with *two* unknowns. We start with two equations, for instance, the first and the second equation:

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s. \end{aligned}$$

This can be easily solved as, by subtracting 2 times the second equation from the first equation, i.e.,

$$7 - 5t = -6 \implies t = \frac{11}{5},$$

$$s = \frac{1+t}{2} = \frac{8}{5}.$$

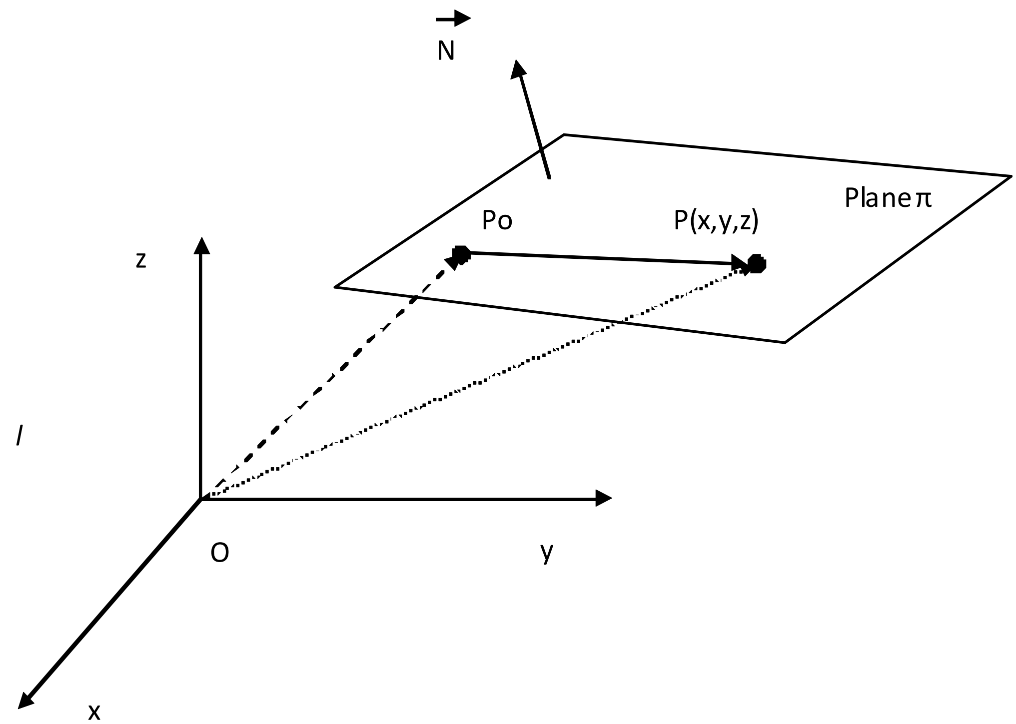
We need to verify that the solution, $t = \frac{11}{5}$, $s = \frac{8}{5}$, from the first two equations (1) & (2), satisfies the third equation (3). So

$$\begin{aligned} LHS\ of\ (3) &= 4 - t = 4 - \frac{11}{5} = \frac{9}{5} \\ RHS\ of\ (3) &= -3 + 4s = -3 + 4\left(\frac{8}{5}\right) = \frac{17}{5}. \end{aligned}$$

Apparently,

$$t = \frac{11}{5}, s = \frac{8}{5}$$

is not a solution of the entire system (1)-(3). We thus conclude that these two line cannot possibly intersect. Answer: skew lines



B) Equations of Plane.

Definition. Any vector that is perpendicular to a plane is called a normal vector to the plane.

There are two normal directions (opposite to each other) to a given plane. For any given vector $\vec{n}$, there are infinite many parallel planes that are all having $\vec{n}$ as their normal vector. If we also know a point on the plane, then, this plane is uniquely determined. In other words, a plane π can be determined by a point $P_0\left(x_0, y_0, z_0\right)$ on the plane and a vector as its normal vector $\vec{n} = \langle A, B, C \rangle$.

For any point $P\left(x, y, z\right)$, if this point P is on the plane π, then the line segment P_0P entirely lies on the plane. Consequently, vector

$$\overrightarrow{P_0P} = \vec{r} - \overrightarrow{r_0} = \langle x - x_0, y - y_0, z - z_0 \rangle, \quad \text{where } \vec{r} = \langle x, y, z \rangle, \ \overrightarrow{r_0} = \langle x_0, y_0, z_0 \rangle,$$

is perpendicular to the normal vector $\vec{n}$. On the other hand, if P is off the plane π, then $\vec{r} = \overrightarrow{P_0P}$ is not perpendicular to $\vec{n}$. We conclude that

$$P \in \pi \quad (P \textit{ belongs to } \pi) \iff \overrightarrow{P_0P} \cdot \vec{n} = 0,$$

or

$$(\vec{r} - \overrightarrow{r_0}) \cdot \vec{n} = 0. \qquad \text{(Vector Equation)}$$

We call it vector equation of the plane π.In terms of components,

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle A, B, C \rangle = 0.$$

We obtain scalar form of equation of plane π :

$$A (x - x_0) + B (y - y_0) + C (z - z_0) = 0, \qquad \text{(Scalar Equation)}$$

or

$$Ax + By + Cz + D = 0. \qquad \text{(Linear Equation)}$$

In $3D$ spaces, any linear equation as above represents a plane with a normal vector $\vec{n} = \langle A, B, C \rangle$. (In 2D, any linear equation is a straight line.)

Example 4.4. Find the equation of the plane passing through $P_0 (2, 4, -1)$ having a normal vector $\vec{n} = \langle 2, 3, 4 \rangle$.

Solution: $A = 2, B = 3, C = 4$.The equation is

$$2 (x - 2) + 3 (y - 4) + 4 (z + 1) = 0,$$

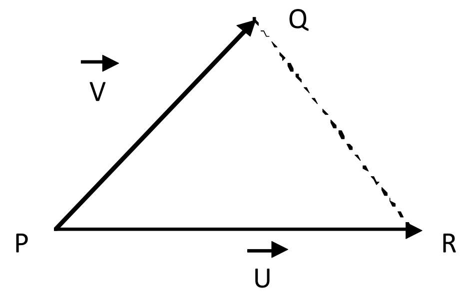
or

$$2x + 3y + 4z - 12 = 0.$$

Example 4.5. Find the equation of the plane passing through $P (1, 3, 2)$, $Q (3, -1, 6)$, $R (5, 2, 0)$.

Solution: Let

$$\vec{u} = \overrightarrow{PR} = \langle 5, 2, 0 \rangle - \langle 1, 3, 2 \rangle = \langle 4, -1, -2 \rangle$$
$$\vec{v} = \overrightarrow{PQ} = \langle 3, -1, 6 \rangle - \langle 1, 3, 2 \rangle = \langle 2, -4, 4 \rangle$$



The vector

$$\vec{u}\times\vec{v}=\begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & -4 & 4 \end{vmatrix}$$
$$=\begin{vmatrix} -1 & -2 \\ -4 & 4 \end{vmatrix}\vec{i}-\begin{vmatrix} 4 & -2 \\ 2 & 4 \end{vmatrix}\vec{j}+\begin{vmatrix} 4 & -1 \\ 2 & -4 \end{vmatrix}\vec{k}$$
$$=-12\vec{i}-20\vec{j}-14\vec{k}$$
$$=-2\left(6\vec{i}+10\vec{j}+7\vec{k}\right)$$

is perpendicular to both $\vec{u}$ and $\vec{v}$.Thus,

$$\vec{n}=\left(6\vec{i}+10\vec{j}+7\vec{k}\right)$$

is perpendicular to π. Now, we take this normal vector and one point $P(1,3,2)$ (you may choose Q or R,instead), and the equation is

$$6(x-1)+10(y-3)+7(z-2)=0$$

or

$$6x+10y+7z-50=0.$$

Note that if we chose $Q(3,-1,6)$ as the known point, then the equation would be

$$6(x-3)+10(y+1)+7(z-6)=0$$

or

$$6x+10y+7z-50=0.$$

Example 4.6. Find the intersection, if any, of the line

$$x = 2 + 3t,\ y = -4t,\ z = 5 + t$$

and the plane

$$4x + 5y - 2z = 18.$$

Solution: We need to solve the system of all four equations

$$\begin{aligned} x &= 2 + 3t \\ y &= -4t \\ z &= 5 + t \\ 4x + 5y - 2z &= 18 \end{aligned}$$

for x, y, z, t. To this end, we substitute the first three equations into the last one. This leads to

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18,$$

which can be simplified to

$$-10t - 2 = 18.$$

So

$$t = -2,$$

and

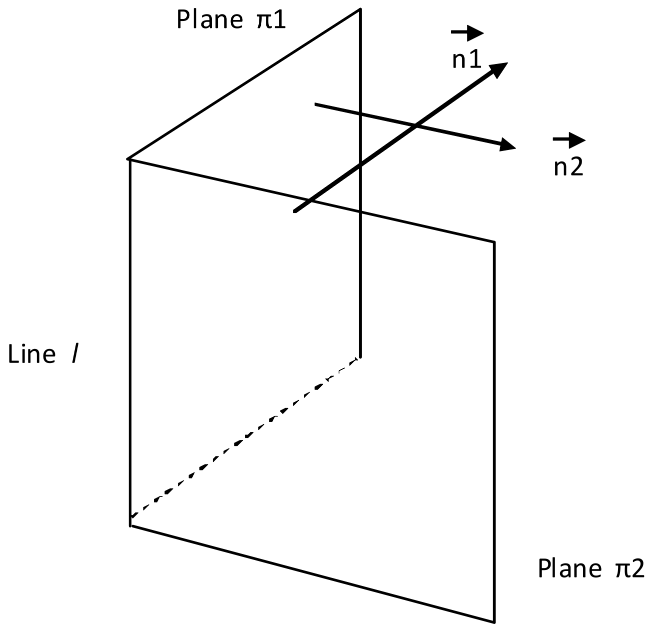
$$\begin{aligned} x &= 2 + 3t = -4 \\ y &= -4t = 8 \\ z &= 5 + t = 3. \end{aligned}$$

Answer: the intersection is $(-4, 8, 3)$.

Example 4.7. Given two planes

$$\begin{aligned} \pi_1 &: \ x + y + z = 1 \\ \pi_2 &: x - 2y + 3z = 1. \end{aligned}$$

Find (a) the line of intersection, and (b) the angle between two planes.



Solution: Plane π_1 and plane π_2 have normal vectors $\vec{n}_1$ and $\vec{n}_2$,respectively, as

$$\vec{n}_1 = \langle 1, 1, 1\rangle$$
$$\vec{n}_2 = \langle 1, -2, 3\rangle .$$

The line is on both planes and thus is perpendicular to both normal vectors. The direction of the line is

$$\begin{aligned}\vec{v} = \vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \vec{k} \\ &= 5\vec{i} - 2\vec{j} - 3\vec{k}.\end{aligned}$$

To find the equation of the line, we also need a point on the line, i.e., on both planes. So we look for one solution to the system

$$x + y + z = 1$$
$$x - 2y + 3z = 1.$$

This system has infinite many solutions (why). Since we only need one solution, we set $z = 0$ to reduce the system to

$$\begin{aligned} x + y &= 1 \\ x - 2y &= 1. \end{aligned}$$

Subtracting the second equation from the first, we find

$$\begin{aligned} 3y &= 0 \implies y = 0 \\ x &= 1. \end{aligned}$$

So $P(1,0,0) \in l$. The equation of the line, in parametric form, is

$$\begin{aligned} x &= 1 + 5t \\ y &= -2t \\ z &= -3t. \end{aligned}$$

Solution #2: Another way to find the equation of this line is to solve the system

$$\begin{aligned} x + y + z &= 1 \\ x - 2y + 3z &= 1 \end{aligned}$$

directly in terms of z. In other words, we choose z as parameter. To this end, we subtract the second equation from the first one to get

$$3y - 2z = 0 \implies y = \frac{2}{3}z.$$

Substituting this into plane π_1 :

$$x + \left(\frac{2}{3}z\right) + z = 1 \implies x = 1 - \frac{5}{3}z,$$

we obtain the equation of the line

$$\begin{aligned} x &= 1 - \frac{5}{3}z \\ y &= \frac{2}{3}z \\ z &= z. \end{aligned}$$

Note that z is basically a free variable. If we set $z=-3t$, this becomes

$$\begin{aligned} x &= 1+5t \\ y &= -2t \\ z &= -3t \end{aligned}$$

which is identical to what we got earlier.

(b) The angle between two planes is the same as the angle between their normal vectors. So

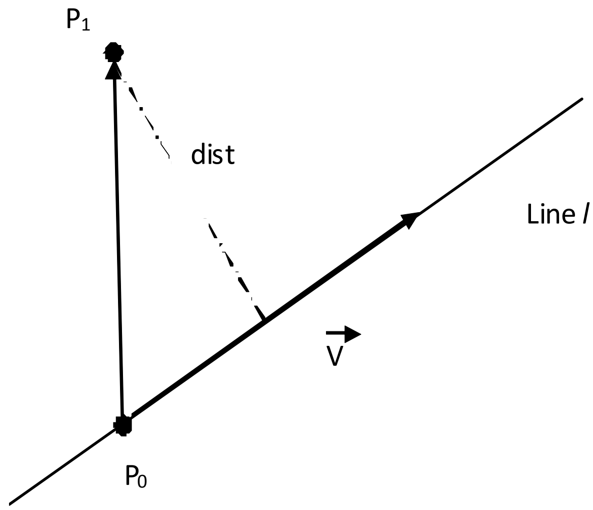
$$\begin{aligned} \cos\theta &= \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{\langle 1,1,1\rangle\cdot\langle 1,-2,3\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,-2,3\rangle|} = \frac{1-2+3}{\sqrt{3}\sqrt{1+4+9}} = \frac{2}{\sqrt{3}\sqrt{14}} \\ \theta &= \arccos\left(\frac{2}{\sqrt{3}\sqrt{14}}\right) = 1.2571(rad) = 1.2571\frac{180}{\pi}\,(\deg) = 72^0. \end{aligned}$$

Practical advice in finding equations of lines or planes:

Regardless what information a problem provides, we always look for a point and a direction. That would be sufficient to solve the problem. More precisely

(1) If we want equations of a line, then we look for one point on the line and a vector PARALLEL to the line.

(2) If we want an equation of a plane, then we look for one point on the plane and a vector PERPENDICULAR to the plane. Cross product may be used to create a vector perpendicular given two vectors.



C) Distance between points, lines and planes:

Let S and T be two sets of points. Then

$$dist\,(S,T) = \min\left\{dist\,(P,Q) \mid P \in S,\ Q \in T\right\}.$$

In other words, the distance between two sets is defined as the smallest distance between two points from different sets.

1) Distance between a point $P_1\,(x_1, y_1, z_1)$ and the line l:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

Pick a point on the line, say $P_0\,(x_0, y_0, z_0)$, and a direction (unit vector) of the line

$$\vec{v} = \frac{1}{\sqrt{a^2+b^2+c^2}}\,\langle a, b, c\rangle\,.$$

Then the cross product

$$\left(\overrightarrow{P_0P_1}\right) \times \vec{v}$$

by definition, has the length

$$dist\,(P_1, l) = \left|\left(\overrightarrow{P_0P_1}\right) \times \vec{v}\right| = \left|\overrightarrow{P_0P_1}\right| \sin\theta.$$

Example 4.8. Find the distance from $P_1\,(1, -2, 1)$ to the line l :

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 - 3t \\ z &= 4t. \end{aligned}$$

Solution. $P_0\,(1, 2, 0)$ is a point on l, and

$$\overrightarrow{P_0P_1} = \langle 1, -2, 1\rangle - \langle 1, 2, 0\rangle = \langle 0, -4, 1\rangle\,.$$

A unit direction of the line is

$$\vec{v} = \frac{1}{\sqrt{29}}\,\langle 2, -3, 4\rangle$$

So

$$\begin{aligned} \left(\overrightarrow{P_0P_1}\right) \times \vec{v} &= \frac{1}{\sqrt{29}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -4 & 1 \\ 2 & -3 & 4 \end{vmatrix} \\ &= \frac{1}{\sqrt{29}} \left(\begin{vmatrix} -4 & 1 \\ -3 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & -4 \\ 2 & -3 \end{vmatrix} \vec{k} \right) \\ &= \frac{1}{\sqrt{29}} \left(-13\vec{i} + 2\vec{j} + 8\vec{k} \right), \end{aligned}$$

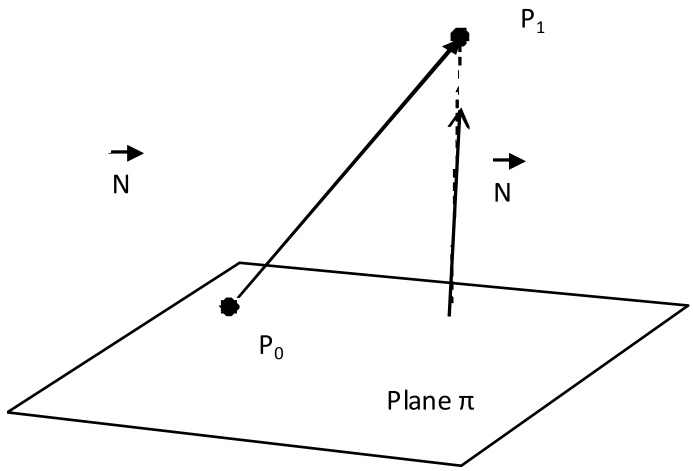
and

$$dist\,(P_1, l) = \left|\left(\overrightarrow{P_0P_1}\right) \times \vec{v}\right| = \frac{\sqrt{13^2 + 4 + 64}}{\sqrt{29}} = \sqrt{\frac{237}{29}} = 2.86.$$

(2) Distance from a point $P_1\,(x_1, y_1, z_1)$ to a plane $\pi : Ax + By + Cz + D = 0$.

Pick any point $P_0\,(x_0, y_0, z_0)$ on the plane, i.e., (x_0, y_0, z_0) solves

$$Ax_0 + By_0 + Cz_0 + D = 0,$$



then the distance is the absolute value of the dot product of $\overrightarrow{P_0P_1}$ and normal direction

$$\vec{n} = \frac{1}{\sqrt{A^2+B^2+C^2}}\langle A, B, C\rangle$$

$$\begin{aligned} dist\,(P_0, \pi) &= \left|\left(\overrightarrow{P_0P_1}\right)\cdot\vec{n}\right| \\ &= \left|\frac{\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle\cdot\langle A, B, C\rangle}{\sqrt{A^2+B^2+C^2}}\right| \\ &= \left|\frac{A\,(x_1-x_0)+B\,(y_1-y_0)+C\,(z_1-z_0)}{\sqrt{A^2+B^2+C^2}}\right| \\ &= \left|\frac{Ax_1+By_1+Cz_1-(Ax_0+By_0+Cz_0)}{\sqrt{A^2+B^2+C^2}}\right| \\ &= \left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|. \end{aligned}$$

Example 4.9. Find the distance from $P\,(1, 2, 3)$ to the plane$\pi : 2x - y + 3z = 4$.

Solution: We rewrite plane π in the standard form as

$$2x - y + 3z - 4 = 0.$$

So

$$\begin{aligned} dist\,(P,\pi) &= \left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right| \\ &= \left|\frac{2x_1-y_1+3z_1-4}{\sqrt{2^2+1^2+4^2}}\right| \\ &= \left|\frac{2-2+9-4}{\sqrt{2^2+1^2+4^2}}\right| = 1.09. \end{aligned}$$

(3) Distance between two lines.

We first fine the plane π containing line l_1 and being parallel to l_2. If $\vec{v}_1$ and $\vec{v}_2$ are directions of l_1 and l_2,respectively. Then,

$$\vec{n} = \vec{v}_1 \times \vec{v}_2$$

is a normal vector to the plane π.One may pick any point P_0 on l_1 and this normal vector to obtain the equation of π. Pick any point P_1 on line l_2,and

$$dist\,(l_1,l_2) = dist\,(P_1,\pi) \tag{A}$$

Another approach is to use projection. Pick one point from each line, say $P_0 \in l_1,\ P_1 \in l_2$. Then,

$$dist\,(l_1,l_2) = \left|\text{Proj}_{\vec{n}}\left(\overrightarrow{P_0P_1}\right)\right| = \frac{\left|\left(\overrightarrow{P_0P_1}\right)\cdot\vec{n}\right|}{|\vec{n}|} \tag{B}$$

Example 4.10. Consider two skewed lines in Example 9.5.3.

$$\begin{aligned} l_1 &: \ x = 1+t,\ y = -2+3t,\ z = 4-t \\ l_2 &: \ x = 2t,\ y = 3+t,\ z = -3+4t. \end{aligned}$$

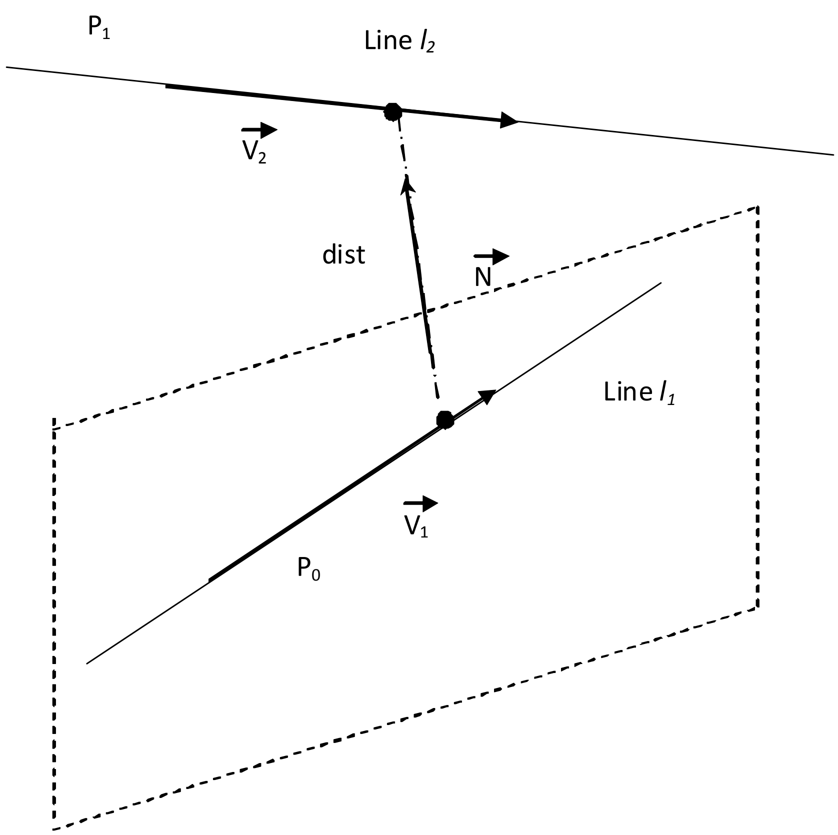
Find their distance.

Solution. We first use method (A). The normal to the plane π containing l_1 and parallel to l_2 is

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 =$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$$= 13\vec{i} - 6\vec{j} - 5\vec{k}.$$


P1
Line l2
V2
dist
N
Line l1
V1
P0

Obviously, $P_0\,(1,-2,4) \in l_1$. So equation of π is

$$13\,(x-1) - 6\,(y+2) - 5\,(z-4) = 0.$$

Since $P_1\,(0,3,-3) \in l_2$, formula (A) and the distance formula lead to

$$\begin{aligned} dist\,(l_1,l_2) &= dist\,(P_1,\pi) \\ &= \left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right| \\ &= \left|\frac{13\,(0-1)-6\,(3+2)-5\,(-3-4)}{\sqrt{13^2+36+25}}\right| \\ &= 0.527\,5 \end{aligned}$$

Let now solve the same problem using formula (B). Note that

$$\begin{aligned} \overrightarrow{P_0P_1} &= \langle 0,3,-3\rangle - \langle 1,-2,4\rangle = \langle -1,5,-7\rangle \\ \vec{n} &= 13\vec{i} - 6\vec{j} - 5\vec{k}. \end{aligned}$$

So

$$dist\,(l_1,l_2) = \frac{\left|\left(\overrightarrow{P_0P_1}\right)\cdot\vec{n}\right|}{|\vec{n}|} = \left|\frac{-1\cdot 13 + 5\cdot(-6) + (-7)\cdot(-5)}{\sqrt{13^2+36+25}}\right| = 0.527\,5.$$

(4) Distance between a line l and a plane π.

Pick any point P_0 on the line and the distance $dist\,(l,\pi)$ between line l and plane π is the distance from P_0 to plane π :

$$dist\,(l,\pi) = dist\,(P_0,\pi)\,,\quad P_0 \in l.$$

Homework:

1. Determine whether each statement is true or false.

   (a) Two lines parallel to a third line are parallel.

   (b) Two lines perpendicular to a third line are parallel.

   (c) Two planes parallel to a third plane are parallel.

   (d) Two planes perpendicular to a third plane are parallel.

(e) Two lines parallel to a plane are parallel.

(f) Two lines perpendicular to a plane are parallel.

(g) Two planes parallel to a line are parallel.

(h) Two planes perpendicular to a line are parallel.

(i) Two planes are either intersect or are parallel.

(j) Two lines are either intersect or are parallel.

(k) A plane and a line are either intersect or are parallel.

2. Find parametric equation and symmetric equation of line.

   (a) The line through the point $(1, 0, -3)$ and parallel to $\langle 2, -4, 5 \rangle$.

   (b) The line through the point the origin and parallel to the line $x = 2t,\ y = 1 - t, z = 4 + 3t$.

   (c) The line through $(1, 1, 6)$ and perpendicular to the plane $x + 3y + z = 5$.

   (d) The line through $(2, 1, 1)$ and perpendicular to $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{k}$.

   (e) The line of intersection of the planes $x + 2y + z = 1$ and $x + y = 0$.

3. Find equation of plane.

   (a) The plane through $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 0)$.

   (b) The plane through $(5, 1, 0)$ and parallel to two lines $x = 2 + t,\ y = -2t,\ z = 1$ and $x = t,\ y = 2 - t,\ z = 2 - 3t$.

   (c) The plane through $(-1, 2, 1)$ and contains the line of intersection of two planes $x + y - z = 2$ and $2x - y + 3z = 1$.

   (d) The plane through $(-1, 2, 1)$ and perpendicular to the line of intersection of two planes $x + y - z = 2$ and $2x - y + 3z = 1$.

   (e) The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$, and is perpendicular to the plane $x + y - 2z = 1$.

4. Determine whether two lines are parallel, skew, or intersecting. If they intersect, find the point of intersection.

(a) $L_1 : x = 1+2t,\ y = 3t,\ z = 2-t;\ L_2 : x = -1+s,\ y = 4+s,\ z = 1 + 3s.$

(b) $L_1 : \dfrac{x-2}{2} = \dfrac{y-3}{2} = \dfrac{z-2}{-1};\ L_2 : \dfrac{x-2}{1} = \dfrac{y-6}{-1} = \dfrac{z+2}{3}$

5. (Optional) Find distance.

(a) Distance from $(1, 0, -1)$ to the line $x = 5 - t,\ y = 3t,\ z = 1 + 2t.$

(b) Distance from $(3, -2, 7)$ to the plane $4x - 6y + z = 5.$

(c) Distance between two planes $3x+6y-9z = 4$ and $x+2y-3z = 2.$