Section 1.4. Power Series

**Definition.** The function defined by

\[ f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \]  \hspace{1cm} (1)

\[ = c_0 + c_1 (x - a) + c_2 (x - a)^2 + c_3 (x - a)^3 + \ldots \]

is called a power series centered at \( x = a \) with coefficient sequence \( \{c_n\}_{n=0}^{\infty} \). The domain of this function consists of all real numbers \( x \) such that the power series is convergent, i.e.,

\[ D(f) = \left\{ x : \sum_{n=0}^{\infty} c_n (x - a)^n \text{ is convergent} \right\}. \]

**Example 4.1.** Some examples of power series:

\[ f(x) = \sum_{n=0}^{\infty} (n!) x^n = 1 + x + (2!) x^2 + (3!) x^3 + (4!) x^4 + \ldots \]
\[ c_n = n!, \ a = 0 \]

\[ g(x) = \sum_{n=1}^{\infty} \frac{(x - 3)^n}{n} = \frac{(x - 3)}{1} + \frac{(x - 3)^2}{2} + \frac{(x - 3)^3}{3} + \ldots \]
\[ c_n = \frac{1}{n}, \ a = 3 \]

\[ h(x) = \sum_{n=0}^{\infty} \frac{(x + 1)^n}{n!} = 1 + (x + 1) + \frac{(x + 1)^2}{2!} + \frac{(x + 1)^3}{3!} + \ldots \]
\[ c_n = \frac{1}{n!}, \ a = -1. \]

- **Interval of Convergence**

We now use Ratio Test to study domain of power series (1). To this end, we write, using the standard series notation

\[ \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n (x - a)^n \]
with 
\[ a_n = c_n (x - a)^n \]
and compute 
\[ a_{n+1} = c_{n+1} (x - a)^{n+1} \]
and 
\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{c_{n+1} (x - a)^{n+1}}{c_n (x - a)^n} \right| = \left| \frac{c_{n+1}}{c_n} \right| |x - a|. \]
So 
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \left| \frac{c_{n+1}}{c_n} \right| |x - a| \right) = \left( \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| \right) |x - a|. \]
Define a number \( R \) by 
\[ R = \frac{1}{\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|} = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|. \]
(2)
(so that 
\[ \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \frac{1}{R} \].
We call \( R \) the **Radius of Convergence** for the power series. It follows that 
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left( \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| \right) \cdot |x - a| = \frac{|x - a|}{R}. \]
By the Ratio Test, the power series is absolutely convergent if 
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x - a|}{R} < 1 \text{ or } |x - a| < R, \]
and it is divergent if 
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x - a|}{R} > 1 \text{ or } |x - a| > R. \]
In other words, if we draw a circle centered at \( a \) with radius \( R \), then the power series is absolutely convergent inside interval \((a - R, a + R)\) and divergent outside the interval.
absolutely convergent inside (a-R, a+R) 

\[ a-R \quad \boxed{a} \quad a+R \]

divergent outside

uncertain at endpoints: 

at each endpoint a-R or a+R,

it could be convergent or divergent

However, at this point we know nothing about what would happen at the endpoints \( x = a - R \) and \( x = a + R \).

**Example 4.2.** Find radius of convergence for the series in Example 4.1:

- \( f(x) = \sum_{n=0}^{\infty} (n!) x^n \), \( c_n = n! \), \( a = 0 \)
- \( g(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \), \( c_n = \frac{1}{n} \), \( a = 3 \)
- \( h(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} \), \( c_n = \frac{1}{n!} \), \( a = -1 \)

Solution: (a) \( c_n = n! \), \( c_{n+1} = (n+1)! \), so

\[
R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0.
\]

(b) \( c_n = \frac{1}{n} \), \( c_{n+1} = \frac{1}{n+1} \),

\[
R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)}{n} = 1.
\]

(c) \( c_n = \frac{1}{n!} \), \( c_{n+1} = \frac{1}{(n+1)!} \),

\[
R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \lim_{n \to \infty} (n+1) = \infty.
\]
We summarize the above analysis as follows.

**Theorem.** Consider the power series

\[ f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \]

and its Radius of Convergence, if exists,

\[ R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|. \]

There are three possibilities:

1. If \( R = 0 \), then the power series is convergent only at \( x = a \);
2. If \( R = \infty \), then the power series is absolutely convergent for all \( x \), and thus \( f(x) \) is defined everywhere;
3. If \( 0 < R < \infty \), then the power series is absolutely convergent for \( |x - a| < R \), i.e.,
   \[ a - R < x < a + R, \]
   and is divergent for \( |x - a| > R \). It is not clear what happens when \( |x - a| = R \) or \( x = a \pm R \).

In short, the domain of the power series (1) is an interval with endpoints \( x = a \pm R \). This interval could be an open interval, a closed interval, or half open half closed interval, and is called Interval of Convergence.

**Finding Interval of Convergence:**

Step #1: find the radius of convergence \( R \), and then write down an interval of the form \( \{a - R, a + R\} \)

Step #2: check convergence of the series \( f(a - R) \) and \( f(a + R) \) at two endpoints \( x = a - R \) and \( x = a + R \)

**Example 4.3.** Find interval of convergence for

(a) \( f(x) = \sum_{n=0}^{\infty} \frac{3^n x^n}{\sqrt{n+1}} \);

(b) \( g(x) = \sum_{n=0}^{\infty} \frac{n (x + 2)^n}{5^n} \).

**Solution:** (a) We know that \( a = 0 \), and we first find its radius of convergence. To do so, write down

\[ c_n = \frac{3^n}{\sqrt{n+1}}, \quad c_{n+1} = \frac{3^{(n+1)}}{\sqrt{(n+1) + 1}}, \]
and calculate
\[ R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \left( \frac{3^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n+2}}{3^{(n+1)}} \right) = \lim_{n \to \infty} \left( \frac{1}{3} \frac{\sqrt{n+2}}{\sqrt{n+1}} \right) = \frac{1}{3}. \]

We then write down the interval of the form
\[ \{a - R, a + R\} = \left\{ -\frac{1}{3}, \frac{1}{3} \right\} \]
that only indicates the interval of convergence is an interval with endpoints \(-\frac{1}{3}\) and \(\frac{1}{3}\). It could be of any type: open interval, or closed interval, or a half open interval. Thus,

\[ \text{interval of convergence has the form } \left\{ -\frac{1}{3}, \frac{1}{3} \right\}, \]

where the type of interval needs to be determined. Next, we check each endpoints.

At the left endpoint \(x = -\frac{1}{3}\),
\[ f \left( -\frac{1}{3} \right) = \sum_{n=0}^{\infty} 3^n \left( -\frac{1}{3} \right)^n \frac{1}{\sqrt{n+1}} = \sum_{n=0}^{\infty} 3^n \left( -\frac{1}{3} \right)^n \frac{1}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n \frac{1}{\sqrt{n+1}}. \]

By Alternating Series Test, we find it convergent (but not absolutely convergent).

At the right endpoint \(x = \frac{1}{3}\),
\[ f \left( \frac{1}{3} \right) = \sum_{n=0}^{\infty} 3^n \left( \frac{1}{3} \right)^n \frac{1}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}. \]

This is a divergent p-series. We thus conclude that
\[ \text{interval of convergence} = \left[ -\frac{1}{3}, \frac{1}{3} \right). \]

(b) We follow the same process to study
\[ g(x) = \sum_{n=0}^{\infty} \frac{n (x + 2)^n}{5^n}. \]
In this case, \( a = -2 \),

\[
c_n = \frac{n}{5^n}, \quad c_{n+1} = \frac{n+1}{5^{n+1}},
\]

\[
R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \frac{n}{5^n} \cdot \frac{5^{n+1}}{n+1} = \lim_{n \to \infty} \frac{5n}{n+1} = 5.
\]

So the interval of convergence has the form

\[
\{ a - R, a + R \} = \{-7, 3\}.
\]

We next check convergence at endpoints. At \( x = -7 \),

\[
g(-7) = \sum_{n=0}^{\infty} \frac{n(-7+2)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(-1)^n 5^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n n.
\]

This series is divergent (by Divergence Test). At \( x = 3 \),

\[
g(3) = \sum_{n=0}^{\infty} \frac{n(3+2)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(5)^n}{5^n} = \sum_{n=0}^{\infty} n
\]

is obviously divergent. We conclude that

\[
\text{interval of convergence} = (-7, 3).
\]

- Derivatives and Integrals

Suppose that a power series

\[
f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n
\]

has the radius of convergence \( R \). Then

**Theorem.** Inside the interval of convergence, i.e.,

\[
|x - a| < R
\]

we have

\[
\frac{df}{dx} = \sum_{n=0}^{\infty} (c_n (x - a)^n)' = \sum_{n=1}^{\infty} nc_n (x - a)^{n-1}, \quad \frac{d^2 f}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n (x - a)^{n-2}, ...
\]
and

\[ \int f(x) \, dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}, \]  
\( C \) is an arbitrary constant.

If \( c \) and \( d \) are inside the interval of convergence, i.e., \(|c-a|, |d-a| < R\), then

\[ \int_c^d f(x) \, dx = \sum_{n=0}^{\infty} c_n \int_c^d (x-a)^n \, dx. \]

**Example 4.4.** For each series, find the integral of convergence and derivative and anti-derivative. Find also an explicit formula.

(a) \( f(x) = \sum_{n=0}^{\infty} x^n \),  \( g(x) = \sum_{n=0}^{\infty} 5^{n+1} x^{3n} \)

Solution: (a) Since \( c_n = 1 \), the radius of convergence

\[ R = \lim \left| \frac{c_n}{c_{n+1}} \right| = 1. \]

Within the interval of convergence, i.e., \((-1, 1)\), we have

\[ f'(x) = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=1}^{\infty} nx^{n-1} \]

\[ \int f(x) \, dx = \sum_{n=0}^{\infty} \int x^n \, dx = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C. \]

Recall the geometric series

\[ \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{iff} \quad |r| < 1. \]

Therefore

\[ f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

and

interval of convergence  = \((-1, 1)\)
(b) We first calculate, within the interval of convergence,

\[ g'(x) = \sum_{n=0}^{\infty} (5^{n+1}x^{3n})' = \sum_{n=0}^{\infty} 3n5^{n+1}x^{3n-1} \]

\[ \int g(x) \, dx = \sum_{n=0}^{\infty} \int 5^{n+1}x^{3n} \, dx = \sum_{n=0}^{\infty} \frac{5^{n+1}}{3n+1}x^{3n+1} + C \]

We next rewrite the series as

\[ g(x) = \sum_{n=0}^{\infty} 5^{n+1}x^{3n} = 5 \sum_{n=0}^{\infty} (5)^n \left(x^3\right)^n = 5 \sum_{n=0}^{\infty} (5x^3)^n = 5 \sum_{n=0}^{\infty} y^n \]

where

\[ y = 5x^3. \]

Thus

\[ g(x) = \sum_{n=0}^{\infty} 5^{n+1}x^{3n} = 5 f(y) = \frac{5}{1-y} = \frac{5}{1-5x^3}, \]

it is convergent for \(|y| = |5x^3| < 1\). So \(|x| < 1/\sqrt{5}\) and

\[ \text{Interval of convergence} = \left( -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right). \]

**Example 4.5.** Study the power series

\[ f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}. \]

**Solution:** Recall from Example 4.4 (a)

\[ \int \sum_{n=0}^{\infty} x^n \, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C \]

and

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \]

Thus,

\[ f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \int \frac{1}{1-x} \, dx = -\ln|1-x| + C. \]
Set $x = 0$, we find
\[ 0 = \sum_{n=1}^{\infty} \frac{0^n}{n} = -\ln |1 - 0| + C = C. \]

We conclude
\[ f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln |1 - x| = -\ln (1 - x) \quad \text{for } |x| < 1. \]

**Homework:**

1. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$.

   What can you say about the convergence or divergence of the following series?

   (a) $\sum_{n=0}^{\infty} c_n$
   
   (b) $\sum_{n=0}^{\infty} c_n 8^n$
   
   (c) $\sum_{n=0}^{\infty} c_n (-3)^n$
   
   (d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

2. Find the radius of convergence and interval of convergence of the series.

   (a) $\sum_{n=0}^{\infty} \sqrt{n} x^n$
   
   (b) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$
   
   (c) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$
   
   (d) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
(e) \[\sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n}} (x + 3)^n\]

3. Find derivatives and anti-derivative. Find also the explicit formula.

(a) \( f(x) = \sum_{n=0}^{\infty} (-1)^n 2^n x^n \)

(b) \( g(x) = \sum_{n=0}^{\infty} 3^{n+1} x^{2n} \)