Appendix: Planar Curves

A) Quadratic Curves.
Recall that we have completely classified quadratic curves in 2D:

\[ Ax^2 + By^2 + Cx + Dy + Exy + F = 0. \]

After a rotation for the coordinate system, it reduces to

\[ Ax^2 + By^2 + Cx + Dy + F = 0. \]

By completing squares, it further reduces to several standard forms:

1. If \( AB \neq 0 \) (i.e., \( A \neq 0, B \neq 0 \)), then

\[ A(x-h)^2 + B(y-k)^2 = R. \]

The curve is either an ellipse with the standard form

\[ \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \]

if \( AB > 0 \) (i.e., \( A \) and \( B \) are either both positive or negative),

\[
\begin{tikzpicture}
  \begin{scope}
    \draw[->, >=stealth] (0,0) -- (6,0);
    \draw[->, >=stealth] (0,0) -- (0,4);
    \node at (6,0) [right] {x};
    \node at (0,4) [below left] {y};
    \draw (1,1) ellipse (2 and 1);
  \end{scope}
\end{tikzpicture}
\]

Center \((h,k) = (1,1)\), half horizontal axis \(a = 2\), half vertical axis \(b = 1\)
or a hyperbolic curve with the standard forms

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{(horizontal)}
\]

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = -1 \quad \text{(vertical)}
\]

if \( AB < 0 \) (i.e., \( A \) and \( B \) have the opposite signs).

(2) If \( AB = 0 \), but \( A^2 + B^2 \neq 0 \), then it reduces to

\[
Ax^2 + Cx + Dy + F = 0 \quad \text{(if } B = 0 \text{)} \quad \text{or}
\]

\[
By^2 + Cx + Dy + F = 0 \quad \text{(if } A = 0 \text{)}
\]

and the curve is a parabola with the standard form either

\[
y = A(x - h)^2 + k \quad \text{(opening vertically)}
\]

\[
x = B(y - k)^2 + h \quad \text{(opening horizontally)}.
\]
(3) If $A = B = 0$, then it becomes a linear equation and the curve is a straight line.

B) Polar Coordinate Systems

The polar coordinate system consists of the origin $O$, the rotating ray or half line from $O$ with unit tick. A point $P$ in the plane can be uniquely described by its distance to the origin $r = dist(P, O)$ and the angle $\theta$, $0 \leq \theta < 2\pi$. We call $(r, \theta)$ the polar coordinate of $P$. Suppose that $P$ has Cartesian (standard rectangular) coordinate $(x, y)$. Then the relation between two coordinate systems is displayed through the following conversion formula:

Polar Coord. to Cartesian Coord.: \[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

Cartesian Coord. to Polar Coord.: \[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\tan \theta &= \frac{y}{x}
\end{align*}
\]

$0 \leq \theta < \pi$ if $y > 0$, $2\pi \leq \theta < \pi$ if $y \leq 0$. 

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Note that function $\tan \theta$ has period $\pi$, and the principal value for inverse tangent function is
\[-\frac{\pi}{2} < \arctan \frac{y}{x} < \frac{\pi}{2}\]
So the angle should be determined by
\[
\theta = \begin{cases} 
\arctan \frac{y}{x}, & \text{if } x > 0 \\
\arctan \frac{y}{x} + \pi, & \text{if } x < 0 \\
\frac{\pi}{2}, & \text{if } x = 0, \ y > 0 \\
-\frac{\pi}{2}, & \text{if } x = 0, \ y < 0
\end{cases}
\]

**Example A.1.** Find (a) Cartesian Coord. of $P$ whose Polar Coord. is $\left(2, \frac{\pi}{3}\right)$, and (b) Polar Coord. of $Q$ whose Cartesian Coord. is $(-1, -1)$.

Sol. (a)
\[
x = 2 \cos \frac{\pi}{3} = 1,
y = 2 \sin \frac{\pi}{3} = \sqrt{3}.
\]
(b) 

\[ r = \sqrt{1+1} = \sqrt{2} \]

\[ \tan \theta = \frac{-1}{-1} = 1 \implies \theta = \frac{\pi}{4} \text{ or } \theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}. \]

Since \((-1, -1)\) is in the third quadrant, we choose \(\theta = \frac{5\pi}{4}\) so

\[ \left(\sqrt{2}, \frac{5\pi}{4}\right) \text{ is Polar Coord.} \]

Under Polar Coordinate system, the graph of any equation of two variables \(r\) and \(\theta\) is a curve. In particular, there are two families of coordinate curves that form Polar grid:

- \(r = \text{constant}\) --- circle centered at \(O\) with radius \(r\)
- \(\theta = \text{constant}\) --- ray with angle \(\theta\).

**Example A.2.** Consider the curve under Polar Coord.

\[ r = 1 + \sin \theta. \]

(a) Sketch the graph
(b) Find the slope of tangent at \( \theta = \pi/3 \)
(c) Find points where the tangent is horizontal.
Sol. (a) We observe that at \( \theta = 0, r = 1 \). As \( \theta \) increases from 0 to \( \pi/2 \), \( r \) increases from 1 to 2. Then in the second quadrant, as \( \theta \) increases from \( \pi/2 \) to \( \pi \), \( r \) decreases at the same rate as in the first quadrant. In other words, the curve in the second quadrant is symmetric to itself in the first quadrant. If we decreases \( \theta \) from \( \theta = 0 \) to \(-\pi/2 \), \( r \) decreases from \( r = 1 \) to \( r = 0 \). The polar equation may be converted into Cartesian equation as follows: We multiply the polar equation by \( r \) to obtain

\[
\begin{align*}
\quad r^2 &= r + r \sin \theta \\
\quad x^2 + y^2 &= \sqrt{x^2 + y^2} + y.
\end{align*}
\]

From the Cartesian equation, it is obvious that the curve is symmetric about \( y -axis \).

(b) Note that

\[
\begin{align*}
x &= r \cos \theta = (1 + \sin \theta) \cos \theta = \cos \theta + \sin \theta \cos \theta \\
y &= r \sin \theta = (1 + \sin \theta) \sin \theta = \sin \theta + \sin^2 \theta
\end{align*}
\]
So

\[
\text{Slope of tangent } = \frac{dy}{dx} = \frac{dy}{d\theta} = \frac{\cos \theta + 2 \sin \theta \cos \theta}{-\sin \theta + \cos^2 \theta - \sin^2 \theta}.
\]

At \( \theta = \pi/3 \),

\[
\frac{dy}{dx} = \frac{\cos \frac{\pi}{3} + 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}}{-\sin \frac{\pi}{3} + \cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} = -1.
\]

(c) The tangent is horizontal when its slope is zero, i.e., when \( dy/dx = 0 \), or

\[
\cos \theta + 2 \sin \theta \cos \theta = 0 \quad \Rightarrow \quad (1 + 2 \sin \theta) \cos \theta = 0 \quad \Rightarrow \quad 
\cos \theta = 0 \text{ or } \sin \theta = -\frac{1}{2} \quad \Rightarrow \\
\theta = \frac{\pi}{2}, -\frac{\pi}{2} = \frac{7\pi}{6}, -\frac{\pi}{6}.
\]

Note that we should rule out

\[
\theta = -\frac{\pi}{2}
\]

since at this point \( dy/dx \) is undefined (denominator = 0).

**Example A.3.** By converting to Cartesian equations, sketch

\[ r = 2 \cos \theta. \]
Sol. Note that if we directly substitute $r$ and $\theta$ as, using the conversion formula between Polar Coord. and Cartesian Coord.,

$$\sqrt{x^2 + y^2} = 2 \cos \left( \arctan \frac{y}{x} \right),$$

it would lead to a very complicated equation. We shall employ the technique that would replace $r, \theta$ by $x, y,$ without direct substitution. To this end, we multiply by $r$ on the both side of the polar equation

$$r^2 = 2r \cos \theta.$$

We then use the conversion formula to obtain

$$x^2 + y^2 = 2x$$

$$(x - 1)^2 + y^2 = 1.$$

This is a circle centered at $(1, 0)$ with radius $R = 1.$

![Circle graph](image)

C) Parametric Curves

The graph of a function $y = f(x)$ usually is a curve consisting of all points

$$\{(x, f(x)) \mid x \in D(f), \text{ i.e., } \exists \text{ is in the domain of } f\}.$$

However, not all curves can be represented as a graph of a function: a curve is a graph of a function only if it passes "vertical line test", i.e., the curve intersects any vertical line at no more than one point. For instance, the unit circle centered at the origin is not the graph of a single function.
To represent general curves, we introduce another variable $t$, different from both $x$ and $y$, and two functions of $t$, $f(t)$ and $g(t)$. The system of two equations

$$x = f(t), \ y = g(t)$$

(1)

defines a curve as the set

$$\{(x, y) = (f(t), g(t)) \mid t \in D(f) \cap D(g)\}.$$

We call $t$ a parameter and the curve the parametric curve. Equation (1) is called a parametrization for the curve. Note that there are multiple parametrization for the same curve and that the parameter variable $t$ may be chosen for different means. For instance, the above unit circle can be parametrized by the following parametric equations

$$x = \cos t, \ y = \sin t, \ 0 \leq t \leq 2\pi.$$

In this parametrization, $t$ is the angle of the vector $(x, y)$ to positive $x$–axis. The same unit circle may also be represented by

$$x = \cos 2t, \ y = \sin 2t, \ 0 \leq t \leq \pi$$

or

$$x = \sin s, \ y = \cos s \ 0 \leq s \leq 2\pi.$$
In fact, one can easily verify that in either case, \((x, y)\) satisfies
\[ x^2 + y^2 = 1. \]

For any graph \(y = f(x)\), the natural parametrization is
\[ x = t, \quad y = f(t). \]

**Example A.4.** (a) Identify and sketch the parametric curve
\[ x = t^2 - 2t, \quad y = t + 1. \]

(b) parametrize ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

Sol. (a) We can eliminate the parameter \(t\) by solving \(t\) from the second equation
\[ t = y - 1 \]
and substituting it into the first equation
\[ x = (y - 1)^2 - 2(y - 1) \\
= y^2 - 4y + 3 \\
= (y - 2)^2 - 1. \]
This is a parabola with vertex \((-1, 2)\) and opening to the right horizontally.

(b)
\[ x = a \cos t, \quad y = b \sin t. \]

**Example A.5.** Discuss and sketch the parametric curve
\[ x = \sin t, \quad y = \sin^2 t. \]

Solution: We can eliminate \(t\) as
\[ y = \sin^2 t = x^2. \]
So the parametric curve is a part of the parabola. However, since
\[ x = \sin t \]
ranges from \(-1\) to \(1\), the parametric curve is actually a piece the parabola:
\[ y = x^2, \quad -1 \leq x \leq 1. \]
Homework:

1. Determine and sketch curves.
   
   (a) \(2x^2 + 3y^2 + 4x - 18y + 2 = 0\)
   
   (b) \(x^2 - 2y^2 + 4x - 4y + 2 = 0\)
   
   (c) \(3y^2 + 4x - 18y + 2 = 0\)

2. Plot the point whose polar coordinate is given. Then find its Cartesian coordinate.
   
   (a) \( \left( 3, \frac{\pi}{3} \right) \)
   
   (b) \( \left( 2\sqrt{2}, \frac{3\pi}{4} \right) \)

3. Convert from Cartesian coordinate to Polar coordinate.
   
   (a) \( (2\sqrt{3}, -2) \)
   
   (b) \( (-1, -\sqrt{3}) \)

4. Identify the curve by finding a Cartesian equation.
   
   (a) \( r = 2\cos \theta + 4\sin \theta \)
   
   (b) \( r = \tan \theta \sec \theta \)

5. Find the slope of tangent to the polar equation at the given point.
6. Eliminate the parameter to find a Cartesian equation of the curve. Then identify and sketch the curve.

(a) \( x = \sqrt{t}, \ y = 2 - t \)

(b) \( x = 2 \cos \theta, \ y = 3 \sin \theta \)

(c) \( x = \sin t, \ y = \csc t, \ 0 < t < \frac{\pi}{2} \)

7. Find a parametric representation for the curve.

(a) \( x^2 + (y - 1)^2 = 4 \)

(b) General ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).