LESSON 4:
GRAPHS OF SINE AND COSINE WAVES

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1. Introduction

Graphing is one of the most important tools available to scientists, engineers, and other practioners in their efforts to understand functions. A graph characterizes a function's behavior and is an easily remembered visual image of its properties. Data analysis and problem solving frequently involves the use of graphs and the insights they provide. Consequently, a thorough knowledge of the graphs of the elementary trigonometric functions as well as an understanding of techniques used to graph more sophisticated trigonometric waves is instrumental to the student of trigonometry.

This lesson presents the basic graphing strategies used to graph generalized sine and cosine waves from a conceptual point of view[1]. The next section presents the graphs of the elementary sine and cosine functions as functions of the variable t. These basic waves have the property that they deviate from the t-axis by no more than one unit. The introduction of a specific parameter can be used to change this **amplitude**. This is discussed in Section 3. Section 4 examines the effect of changing the period (See Section 5 of Lesson 2.) of sine and cosine waves. Section 5 examines the procedure for shifting a wave along the t-axis. The last section presents techniques for graphing trigonometric waves when all three of these modifications are present in the function.

[1] Graphs of the other four trigonometric functions are examined in the next lesson.

2. Graphs of $y = \sin t$ and $y = \cos t$

Recall that if (x, y) is the point on the unit circle that determines the angle t rad then $\sin t = y$. Table 4.1 lists approximate values of this function at the indicated abscissa. Note the symmetry about the number π. Plotting these points suggests the graph for the function $y = \sin t, 0 \le t \le 2\pi$, shown in Figure 4.1. (Only four points are plotted to avoid a cluttered appearance.) A graphing calculator can be used to generate this *wave* of the sine function if desired.

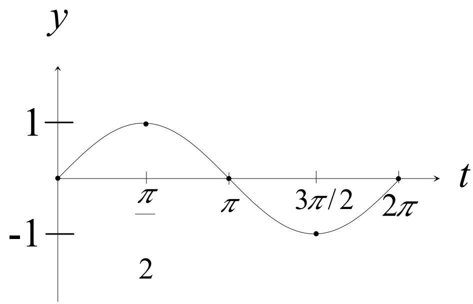
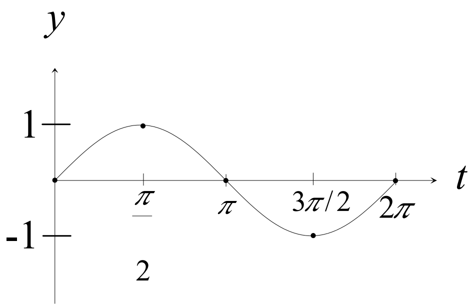


Figure 4.1: Graph of $y = \sin t$ on $[0.2\pi]$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t$	0	$.5$	$.707$	$.866$	1	$.866$	$.707$	$.5$	0
t	$\frac{7\pi}{6}$	$\frac{7\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{9\pi}{4}$	$\frac{11\pi}{6}$	2π	$\ldots$
$\sin t$	$-.5$	$-.707$	$-.866$	-1	$-.866$	$-.707$	$-.5$	0	$\ldots$

Table 4.1: Values of the sine function.



From the sketch of Figure 4.1 reproduced above, it appears that the **amplitude** or maximum deviation (up or down) of the graph from the x-axis is one unit. That is, $-1 \leq \sin t \leq 1$ for all $t \in [0, 2\pi]$. To see that this graphical insight is valid, first note that $\sin \frac{\pi}{2} = 1$. Hence, the graph of y deviates from the x-axis by at least one unit, suggesting that the amplitude is at least one. It remains to be shown that one unit is the maximum deviation of y from the x-axis. Since $\sin t = y$ where y is the second coordinate of a point on the unit circle $x^2+y^2 = 1$, we have $y^2 = 1-x^2 \leq 1 \Longrightarrow |y| \leq 1$. The result

$$|\sin t| = |y| \leq 1$$

follows.

Since the sine function has period 2π, the graph of $y = \sin t$ for all real numbers t has the same appearance as the sketch in Figure 4.1 on the intervals $[-2\pi, 0]$ and $[2\pi, 4\pi]$. More generally, the graph of y has the same appearance as the wave in Figure 4.1 on all intervals of the form

$$[2k\pi, 2(k+1)\pi]$$

where k is any integer. This means that the graph of the sine function can be extended to larger domains simply by redrawing the wave in Figure 4.1 on intervals of this form. Doing so yields the sketch of the sine function given in Figure 4.2.

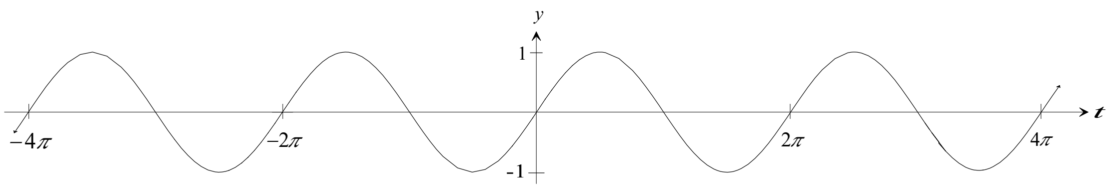


Figure 4.2: Graph of $y = \sin t$.

The strategy of plotting points can be used to verify the accuracy of the graph of $y = \cos t$, $t \in [0, 2\pi]$ shown in Figure 4.3. Notice that this graph appears to be the same as that of a *shifted* sine function. (Compare Figure 4.1 with Figure 4.3.) In fact, as will be verified later (See Equation 5 in Lesson 5.), $\cos t = \sin(\frac{\pi}{2} - t)$ for all real numbers t. This identity establishes that the amplitude of the cosine function is one unit because the sine function (shifted or not) has a maximum deviation of one unit from the x-axis. Of course, the fact that $\cos t$ is the x-coordinate of a point on the unit circle also verifies this fact.

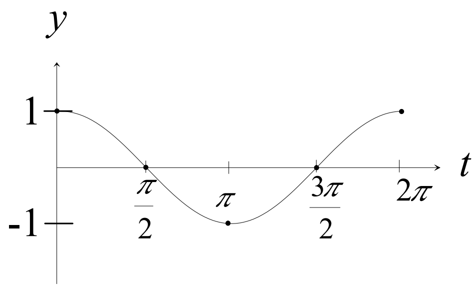


Figure 4.3: Graph of $y = \cos t$ for $t \in [0, 2\pi]$.

As in the case for the sine function the periodicity of the cosine function can be exploited to obtain the wave shown in Figure 4.4.

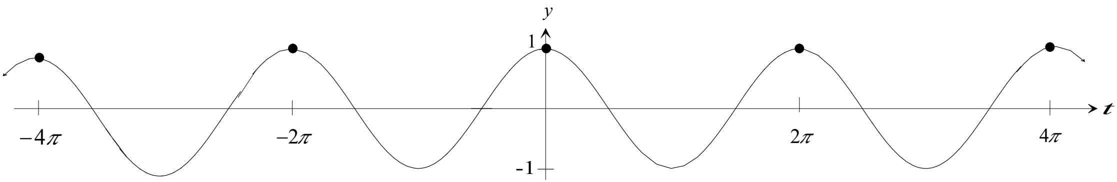


Figure 4.4: Graph of $y = \cos t$.

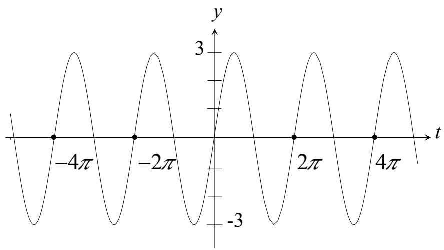
3. Changing the amplitude

Multiplying the sine or cosine function by a constant (positive or negative) changes the maximum height and depth of its graph. This change in amplitude is illustrated in the example below.

Example 1 *Graph the function $y = 3\sin t$.*

Solution: The graph of y is similar to that of the sine function graphed in Figure 4.2 . The only difference is that each value of $\sin t$ is multiplied by 3 as suggested by the values in the table below. In particular, the maximum height of 1 and depth of (-1) of the basic sine function are multiplied by 3. This means that the number 3 is the amplitude of $y = 3\sin t$. The graph of the wave y is given in the Figure (b)elow.

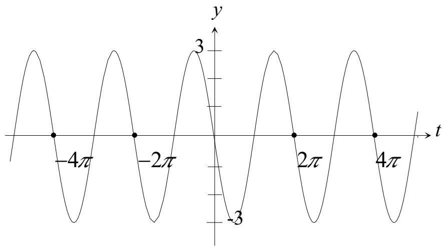
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t$	0	0.5	.707	.866	1	.866	.707	0.5	0
$3\sin t$	0	1.5	2.121	2.598	3	2.598	2.121	1.5	0



Multiplying a sine or cosine function by a negative constant also changes the amplitude by the absolute value of the constant as illustrated in the following example.

Example 2 *Some of the values of* $y(t) = -3\sin t$ *are given the table below while its graph is depicted in the figure. Note that the maximum deviation from the* x*-axis is three units. The student should compare this graph with that for the previous example.*

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$-3\sin t$	0	-1.5	-2.121	-2.598	-3	-2.598	-2.121	-1.5	0

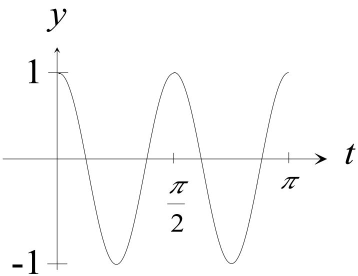


4. Changing the period

The period of the basic sine and cosine waves is 2π. This can be changed by replacing the independent variable t with at for any nonzero number a. This is demonstrated using the function $y = \cos(4t)$ in the following example.

Example 3 *Sketch a graph of two oscillations of* $y = \cos(4t)$.
Solution: As suggested by the table below, the function y *completes one oscillation as the independent variable* t *ranges over the interval* $[0, \pi/2]$. *This means that* y *has period* $\pi/2 = 2\pi/4$. *A sketch of two oscillations of the graph of* y *appears to the right of the table.*

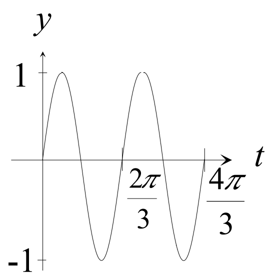
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(4t)$	1	$-.5$	-1	$-.866$	1



Compare the graph above with Figure 4.1 and note the shrinking of the period from 2π to $\pi/2$ that takes place.

The table in the previous example is included to help the student understand the effect of a change in period. He or she will soon realize that it is not necessary to use tabular data to graph a such problem if it is understood that replacing the independent variable t with at (resulting in a change in period) merely shrinks or elongates the wave. In such cases tabular data serves to identify specific points on the curve in much the same way as labels on the coordinate axes. The technique of plotting a sine or cosine wave with adjusted periods without using tabular data is demonstrated in the following example and formalized in the seceding paragraph.

Example 4 *Sketch a graph of two oscillations of* $y(t) = \sin(3t)$.
Solution: The graph of y *must complete one oscillation as* t *ranges over the interval* $[0, 2\pi/3]$ *so* y *has period* $2\pi/3$. *The desired graph is depicted in the figure.*



We now formalize the concept of a change of period of the cosine function. The student should realize that similar reasoning holds for the sine function. For the moment assume a is positive and let $y(t) = \cos(at)$. If $t = 2\pi/a$ then $at = 2\pi$ so the graph of y completes one full wave or oscillation in $2\pi/a$ units (as opposed to 2π units for the unmodified cosine function). Symbolically, substituting $(t + 2\pi/a)$ for t as the argument of y gives

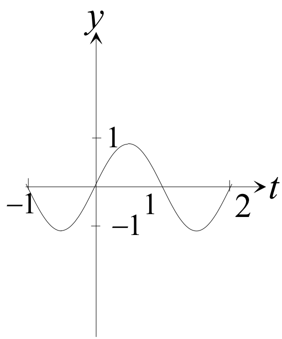
$$y\left(t + \frac{2\pi}{a}\right) = \cos\left[a\left(t + \frac{2\pi}{a}\right)\right] = \cos(at + 2\pi) = \cos(at) = y(t).$$

That is, $y(t + 2\pi/a) = y(t)$ establishing that y is periodic with period $2\pi/a$. These calculations may seem confusing at first but understanding them is crucial to a working knowledge of trigonometry. It may be helpful to fix a number, say $a = 4$, and use a hand calculator (Don't forget to convert the calculator to radian mode!) to compute $\cos(4t)$ and $\cos[4(t + 2\pi/4)]$ for several values of t.

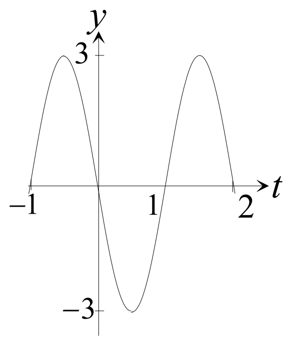
If $0 < a < 1$, then $2\pi/a > 2\pi$ so y has a larger, or longer, period. This has the effect of elongating or stretching the graph of the basic cosine function. The opposite is true if $a > 1$. These observations also hold for the sine function as illustrated in the next example.

Example 5 *Graph* $y(t) = \sin(t/\pi)$ *on the interval* $[-1, 2]$.

Solution: The period of y *is* $2 = 2\pi/\pi$ *so* y *completes one oscillation on the interval* $[0, 2]$ *and 'half' an oscillation on* $[-1, 0]$. *A sketch of the graph of* y *on the given interval is presented in Figure (a) below.*



(a)



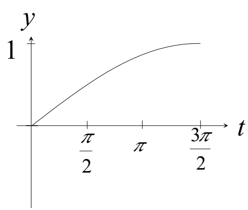
(b)

Example 6 *In view of the previous example plotting the curve* $y(t) = -3\sin(t/\pi)$ *on the interval* $[-1, 2]$ *is relatively easy. There are two differences. First, the amplitude is* 3 *units and the negative sign in front of the* 3 *causes a reflection about the* x*-axis. The graph is given in Figure (b) above.*

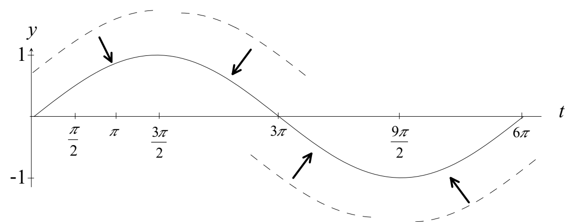
At the beginning of this lesson the periodicity of the sine and cosine functions was used extend the graphs of one oscillation of their waves to larger domains. There are ways to exploit the symmetry of more general sine and cosine waves when constructing their graphs. Note that any such curve is composed of four arcs each with different orientations and locations but having the same shape. (See, for example, the breakdown of the wave in the figure for Example 7.) If one arc of a wave is properly sketched, it can replotted with appropriate changes in locations and orientations to form a complete graph of one oscillation of the function. The periodicity of the function can then be used to construct a complete graph. Consequently, it is usually only necessary to carefully plot these curves for $\frac{1}{4}$ of a full period, and then complete the graph using these symmetry considerations. The next example demonstrates this technique.

Example 7 *Sketch a graph of one wave of* $y = \sin(t/3)$.

Solution: Since $\frac{t}{3} = \frac{1}{3}t$ *the period of* y *is* $\frac{2\pi}{1/3} = 6\pi$. *Thus,* $\frac{1}{4}$ *of a full period of* y *is* $\frac{1}{4}(6\pi) = \frac{3\pi}{2}$. *This suggests that the first arc of the 'first' wave of* y *occurs on the interval* $[0, \frac{3\pi}{2}]$ *with* $y(0) = 0$ *and* $y(\frac{3\pi}{2}) = 1$. *The curve on this interval has the same general appearance as the first arc of a sine wave. A sketch of this arc is given in Figure (a) on the left below. (The reader can plot more points for* y *on the interval* $[0, \frac{3\pi}{2}]$ *to further justify this sketch. The necessity of such calculations should diminish as the level of understanding increases.) Figure (b) illustrates how one complete wave of* y *is constructed by properly locating and orienting copies of this original arc. The arrows and arcs in Figure (b) suggest the appropriate changes in the first needed to form the additional arcs of a wave of* y. *Of course, it is also easy to graph* y *by realizing that the interval* $[0, 6\pi]$ *contains exactly one oscillation of the curve.*



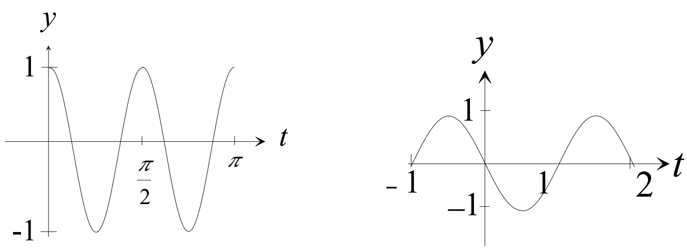
(a)



(b)

Recall that $\sin(-t) = -\sin t$, an identity established in Lesson 2 Section 4. Consequently, replacing t by $-t$ in the sine function introduces a reflection about the x-axis in the graph of the wave. Since $\cos(-t) = \cos(t)$, no reflection is introduced in cosine waves. The next two examples illustrate these observations.

Example 8 *A sketch of the graph of* $y(t) = \cos(-4t)$ *is given in Figure (a) below. Since* $\cos(-4t) = \cos(4t)$, *this graph is the same as the one for Example 3.*



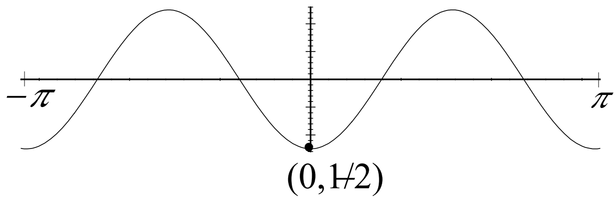
(a) $y(t) = \cos(-4t)$ (b) $y(t) = \sin(-t/\pi)$

Example 9 *Sketch a graph of* $y(t) = \sin(-t/\pi)$ *for* $t \in [-1, 2]$.
Solution: Since $y(t) = -\sin(t/\pi)$ *and* $2 = \frac{2\pi}{1/\pi}$, y *has period* 2. *Figure (b) above depicts the graph of* y. *Compare this graph with the one given in Example 5 and note that the two are reflections of each other about the* x*-axis.*

Example 10 *Sketch a graph of* $y(t) = \sin(-t/3)$
Solution: Since $\sin(-t/3) = -\sin(t/3)$, *the graph of* $y(t)$ *is a reflection about the* x*-axis of the graph given in Example 7.*

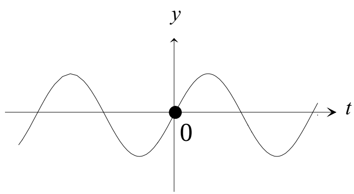
These observations suggest that the period of the functions $\sin(at)$ and $\cos(at)$ is $2\pi/|a|$ for all positive or negative values of a. The final example of this section demonstrates the procedure for graphing a sine or cosine wave that includes modifications in both the amplitude and period.

Example 11 *Graph two complete oscillations of the function* $y(t) = \frac{1}{2}\cos(-2t)$.
Solution: First write $y(t) = -\frac{1}{2}\cos(2t)$ *which suggests that* y *has amplitude* $\frac{1}{2}$ *and period* π*. Hence, the interval* $[-\pi, \pi]$ *contains two complete oscillations of the graph of* y*, as would any interval of length* 2π*. Also, the negative sign introduces a reflection about the* x*-axis. The graph of* $y(t)$ *on* $[-\pi, \pi]$ *is given in the figure below:*

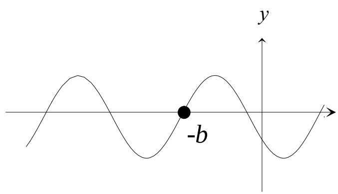


5. Graphs of $y = \sin(t+b)$ and $y = \cos(t+b)$

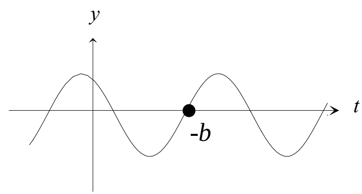
The graphs of the elementary sine and cosine functions are *shifted* or translated to the right or left by adding a constant b to the variable t. This behavior is illustrated in the figures below for the sine function. In general, replacing an independent variable t by $t+b$ (the notation $t \to t+b$ is frequently used) is called a *translation.* Note that $t+b$ has value 0 when $t = -b$. With respect to graphs this means that the role of 0 as the "center of attention" (as indicated by the dots in the graphs) has been replaced by the number $-b$. When used in conjunction with periodic functions, translations are referred to as **phase shifts** because the phase of the basic wave is shifted by a factor of b. If b is positive the curve shifts to the left as demonstrated in Figure b. If b is negative, so that $-b$ is positive, the shift is to the right as shown in Figure c. Graphing shifted sine and cosine waves is not difficult if one is familiar with their unshifted graphs and understands the effect of a translation. The next example demonstrates the translation process.



(a) $y = \sin t$



(b) $y = \sin(t+b)\,,\ b > 0$

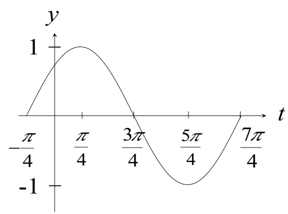


(c) $y = \sin(t+b)\,,\ b < 0$

Example 12 *Sketch one oscillation of the function* $y(t) = \sin(t + \pi/4)$.

Solution: Replacing t *by* $t + \pi/4$ *as the argument of the sine function translates its graph to the* left *by* $\pi/4$ *units as suggested by the table below. The sketch of one oscillation of* y *appears to the right of the table. (Note that such a table of values can be constructed using numbers for the variable* t *that produce quantities at which the trigonometric functions are known. For example, the abscissa* $t = -\frac{\pi}{12}$ *was chosen because it reduced to the familiar argument* $\frac{\pi}{6}$ *under the translation* $t \to t + \frac{\pi}{4}$*.) Once again, the user can avoid tabular data if the notion of a phase shift is understood. Note the relationship between the graph of* y *and the basic sine wave shown in Figure 4.1. Observe that one is just a shifted version of the other. The table of values in this example was provided to help demonstrate the translation process.*

t	$-\frac{\pi}{4}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{4}$
$t + \pi/4$	0	$\frac{\pi}{6}$	$\pi/4$	$\pi/2$
$\sin(t + \pi/4)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1



6. Graphs of $y(t) = A\sin[a(t+b)]$ and $y(t) = A\cos[a(t+b)]$

It is possible to combine the three concepts of amplitude, period, and phase shift into a single problem that requires graphs of functions of the generalized sine and cosine functions $y = A\sin[a(t+b)]$[2] or $y = A\cos[a(t+b)]$. Table 4.2 contains a quick review of the roles of each constant.

$\lvert A\rvert$	amplitude
$\frac{2\pi}{\lvert a\rvert}$	period
b	phase shift

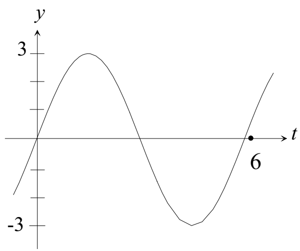
Table 4.2: Effects of constants on generalized sine and cosine functions.

When graphing such functions the reader should exploit his or her knowledge of the effects phase shifts and changes in amplitudes and periods on the graphs of the fundamental sine and cosine waves. Doing so allows the reader to "build" the graphs of generalized functions from less complicated sketches.

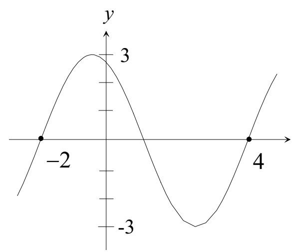
[2]The student should recall that a phase shift is introduced whenever t is replaced by $t+b$. Hence, the function $y = 3\sin[2(t+\pi)]$ has a phase shift of π units to the left while $y = 3\sin[2t+\pi]$ has a phase shift of $\pi/2$ units to the left since $2t+\pi = 2(t+\pi/2)$.

Example 13 *Sketch a graph of the function* $y(t) = 3\sin\left[\frac{\pi}{3}(t+2)\right]$.

Solution: First, observe that the amplitude of y *is* 3. *(At this point the reader should visualize the graph of* $y = 3\sin t$ *which was graphed in Example 1. This is the basic sine wave modified so that the amplitude is* 3.*) Next, note that the period of* $y = 3\sin\left[\frac{\pi}{3}(t+2)\right]$ *is* $\frac{2\pi}{\pi/3} = 6$. *(The reader should modify his or her mental image of the graph in Example 1 so that it completes exactly one oscillation in* 6 *units. The result of this modification is the graph of* $y = 3\cos(\frac{\pi}{3}t)$ *depicted in Figure (a) below.) Finally, the factor* $(t+2)$ *causes a phase shift of* 2 *units to the left. A portion of the graph of* y *appears in Figure (b) below. Observe that the curves in Figures (a) and (b) are the same except for a translation.*

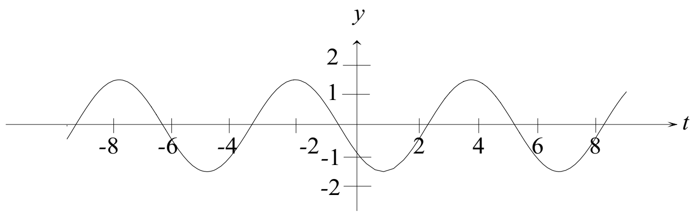


(a)



(b)

Example 14 *Sketch a graph of the function* $y\,(t) = -1.5\cos(\frac{\pi}{3}t - \frac{\pi}{4})$.
Solution: Rewrite $y\,(t) = -1.5\cos[\frac{\pi}{3}(t - \frac{3}{4})]$. *The function* y *has amplitude* 1.5 *units. The negative sign in front of the amplitude introduces a reflection as described in Example 2. (At this point the reader should visualize the graph of* $y_a = -1.5\cos t$, *a basic cosine wave reflected about the* x*-axis and modified so that the amplitude is* 1.5 *units.) The period of* y *is* $\frac{2\pi}{\pi/3} = 6$ *so the wave makes one complete oscillation in* 6 *units. (The reader should modify his or her mental graph so that it completes one oscillation in* 6 *units. The result is the graph of* $y_p = -1.5\cos(\frac{\pi}{3}t)$.*) Finally,* y *has a phase shift of* $-3/4$ *units which effectively shifts the graph of* $y_p = -1.5\cos(\frac{\pi}{3}t)$ $3/4$ *units to the right. A graph of* y *is given below.*



Adding a constant c to, say, $y\,(t) = A\sin[a(t+b)]$ has the effect of raising its graph c units if c is positive or lowering it c units if c is negative. This is demonstrated in Example 6.

7. Exercises

EXERCISE 1. Graph the function $y(t) = 3\cos t$.

EXERCISE 2. Graph $y(t) = \sin(\frac{\pi}{2} - t)$.

EXERCISE 3. Graph $y(t) = \frac{1}{4}\cos\left(\frac{\pi}{6}t\right)$.

EXERCISE 4. Graph $y(t) = \frac{1}{4}\cos\left(\frac{\pi}{6}t + \frac{\pi}{3}\right)$.

EXERCISE 5. Graph $y(t) = -2\cos(\pi t - 0.5\pi)$.

EXERCISE 6. Graph $y(t) = 2\sin(2\pi t - \pi) + 3$.

EXERCISE 7. Find a sine wave whose graph is the one given below.